The local Langlands correspondence

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Or quel che t'era dietro t'e' davanti:
ma perche' sappi che di te mi giova,
un corollario voglio che t'ammanti.
(Dante Alighieri, Paradiso, VIII)

1 Introduction

So far, we have been assuming (with no motivation) that: (1) our base field is **C**, (2) the topology on complex vector spaces is discrete.

At a first glance, this is unsatisfactory if we wish to deal with Galois representations: they have usually coefficients in fields like $\bar{\mathbf{Q}}_\ell$ and, moreover:

Lemma 1.0.1. *Every continuous homomorphism from a profinite group G to $GL_n(\mathbf{C})$ has finite image.*

Proof. Choose an open subgroup $U < G$ with image in the ball $\mathcal{B}$ with center 1 and radius 1/2. The topology on $GL_n(\mathbf{C}) \subset End(\mathbf{C}^n)$ is induced by the norm: $\| T \| = sup_v |Tv|$. Assume $\rho(U) \neq 1$. If, for any T, we can find N such that $\| T^N - 1 \| > 1/2$, we get a contradiction. Indeed, if all the eigenvalues of T are 1, it is clear from the Jordan form of T. If T has an eigenvalue $\lambda \neq 1$, then $|\lambda^N - 1| > 1/2$ for some N.

□

This fact leads us to introduce the Weil group.

2 Weil groups

Fix a separable algebraic closure $\bar{F}$ of F. Let k be the residue field of F, of characteristic p. We have a surjective map $\Gamma_F \to Gal(\bar{k}/k)$ whose kernel is the inertia subgroup I of Γ_F. The arithmetic Frobenius $\sigma \in Gal(\bar{k}/k) \simeq \hat{\mathbf{Z}}$ sends x to x^q and generates topologically $Gal(\bar{k}/k)$.

Definition 2.0.2. The Weil group W_F of F is the inverse image of $< \sigma >$ in Γ_F. Then we have the exact sequence:

$$1 \to I \to W_F \to < \sigma > \to 1 \tag{2.0.1}$$

We topologize W_F by asking that the projection $W_F \to < \sigma > \simeq \mathbf{Z}$ is continuous with respect to the discrete topology on $\mathbf{Z}$ and the induced topology on $I \subset W_F$ equals the profinite topology induced by that of Γ_F.

Remark 2.0.3. Note that the inclusion $W_F \hookrightarrow \Gamma_F$ is continuous and has dense image.

Definition 2.0.4. A n-dimensional representation of W_F is a continuous homomorphism $W_F \to GL_n(\mathbf{C})$ (with discrete topology on the latter group).

Remark 2.0.5. By class field theory, we have an isomorphism: $W_F^{ab} \simeq F^*$.

Remark 2.0.6. If E is a finite extension of F, then W_E is an open subgroup of W_F and $W_F/W_F \simeq Gal(E/F)$ when E/F is Galois.

Remark 2.0.7. Any representation of Γ_F gives by restriction a representation of W_F.

We define a character of W_F by $\omega_s(I) = 1$ and $\omega_s(\sigma) = q^{-s}$, for a complex number s. Note that ω_s is continuous. Then:

Proposition 2.0.8. *Every irreducible representation of W_F is of the form $\rho = \omega_s \otimes \rho'$, with ρ' irreducible of finite image.*

Proof. An irreducible representation ρ of W_F has to be trivial on some subgroup J of I of finite index. Then some twist of ρ by a character trivial on I has finite image. In fact, conjugation by σ induces an automorphism of the finite group I/J, so conjugation by some power σ^t is the identity. Now, $\rho(\sigma^t)$ commutes with $\rho(W_F)$, therefore it acts as a scalar. Choose s such that $(\omega_s \otimes \rho)(\sigma^t) = 1$. Then the kernel of $\omega_s \otimes \rho$ contains J and σ^t and thus has finite index. This proves that the image of $\rho' = \omega_s \otimes \rho$ is finite.

□

Proposition 2.0.9. *A representation of W_F extends to a representation of Γ_F if and only if it has finite image.*

Proof. This follows immediately from lemma 2.0.1 and the density of W_F in Γ_F. □

Theorem 2.0.10. *If $(n, p) = 1$, every irreducible representation ρ of W_F of dimension n over an algebraically closed field is induced by a character of a subgroup of index n of W_F.*

This is deduced from the following facts (see Koch, Inv.Math.40). Let P be the wild inertia subgroup of Γ_F, i.e. the pro-p Sylow of the inertia group I. Then $I/P \simeq \prod_{\ell \neq p} \mathbf{Z}_\ell$ and $\rho|_P$ splits as a sum of 1-dimensional representations.

If ρ is not induced from a proper subgroup, then $\rho|_P$ is irreducible.

We prove the following simple case:

Proposition 2.0.11. *Let $\rho : W_F \to GL_2(\mathbf{C})$ be an irreducible representation, $p \neq 2$. Then ρ is induced from a character of a subgroup of index 2.*

Proof. We can assume that ρ has finite image. Then it factors through some finite extension E/F and $\rho(W_F) = \rho(Gal(E/F))$ by remark 2.1.5. A finite subgroup of $PGL_2(\mathbf{C})$ can be: cyclic, dihedral, A_4, S_4 or A_5. Now, $\rho(Gal(E/F))$ is not cyclic by irreducibility. Recall:

$$1 \subset G_1 \subset I \subset Gal(E/F) \tag{2.0.2}$$

where G_1 is a p-group, $Gal(E/F)/I$ and I/G_1 are cyclic. Since $Gal(E/F)$ is solvable, we can exclude A_5.

If $p \neq 2$, 2.0.2 is incompatible with the structure of A_4 and S_4.

Therefore $Im(\rho)$ is dihedral, so it has a cyclic subgroup of index 2 and ρ is induced from a character of this subgroup.

For the classification of irreducible representations of the dihedral group, see Serre's book, Linear representations of finite groups.

□

3 Weil-Deligne representations

We start with some general facts about ℓ-adic representations of W_F.

Lemma 3.0.12. *There is a unique (up to a constant multiple) nonzero homomorphism $t : I \to \mathbf{Q}_\ell$ if $\ell \neq p$.*

Proof. It suffices to note that $I/P \simeq \prod_{\ell \neq p} \mathbf{Z}_\ell$ □

We fix a choice of t. Note that $t(wgw^{-1}) = \omega_1(w)t(g)$, $w \in W_F, g \in I$.

Theorem 3.0.13. *Let $\rho : W_F \to GL(V)$ be a continuous representation on a finite dimensional V over a finite extension of $\mathbf{Q}_\ell$, $\ell \neq p$. Then there is a nilpotent endomorphism N of V such that $\rho(g) = exp(t(g)N)$ for g in an open subgroup of I.*

For the proof, see Serre,Tate (Ann.Math.88). We call N the monodromy operator of ρ.

Corollary 3.0.14. *With hypothesis as above, if ρ is semisimple some open subgroup of I acts trivially on V.*

Proof. If ρ is irreducible, $N = 0$. A semisimple representation is a direct sum of irreducible representations. □

Definition 3.0.15. Let K be a topological field. An n-dimensional Weil-Deligne representation of W_F over K is a pair (ρ, N), with:

1. $\rho : W_F \to GL(V)$ a representation on a n-dimensional K-vector space V, continuous for the discrete topology of V;

2. N is a nilpotent endomorphism of V, such that $\rho(w)N\rho(w)^{-1} = \omega_1(w)N$ for any $w \in W_F$.

Let $\ell \neq p$ and V be a finite-dimensional vector space over a finite extension E of $\mathbf{Q}_\ell$. We will call a ℓ-adic representation of W_F over E any continuous homomorphism $\rho : W_F \to GL(V)$.

Theorem 3.0.16. *There is a bijection between the set of ℓ-adic representations of W_F over E and the set of W-D representations over E.*

Let us fix an isomorphism $\mathbf{C} \simeq \bar{\mathbf{Q}}_\ell$. Then a semisimple continuous W-D representation $\rho : W_F \to GL_n(\bar{\mathbf{Q}}_\ell)$ is continuous as a complex representation.

Example 3.0.17. The "special" W-D representation $sp(n)$ of W_F: $V = \mathbf{Q}^n$, $\rho(w)e_i = \omega_i(w)e_i$, $Ne_i = e_{i+1}$, $Ne_{n-1} = 0$, where $e_0, ..., e_{n-1}$ is the standard basis.

Define $(\rho, N) \otimes (\rho', N') = (\rho \otimes \rho', N \otimes 1 + 1 \otimes N')$.

Proposition 3.0.18. *Every semisimple indecomposable W-D representation of W_F is of the form $\rho' \otimes sp(n)$, with ρ' irreducible.*

4 L-functions and epsilon factors

For a W-D representation (ρ, N) of W_F we define:

$$L(s, \rho) = det(1 - q^{-s}\rho(\sigma)|ker(N)^I)^{-1} \tag{4.0.3}$$

where σ is the Frobenius as before.

Example 4.0.19. In the case of a 1-dimensional representation, this gives:

$$L(s, \rho) = (1 - \rho(\sigma)q^{-s})^{-1}$$

if ρ is unramified (i.e. $\rho(I) = 1$), and

$$L(s, \rho) = 1$$

if ρ is ramified.

Example 4.0.20. The L-function of the special representation is:

$$L(s, sp(n)) = (1 - q^{-s})^{-1}$$

independently on n.

Definition 4.0.21. The conductor of a continuous representation $\rho : \Gamma_F \to GL(V)$ is the number:

$$c(\rho) = \sum_{i=0}^{\infty} \frac{|G_i|}{|G_0|} codim V^{G_i} \tag{4.0.4}$$

where G_i is the ith ramification subgroup of $G_0 = I$ (cf. Serre, Corps Locaux).

The sum is of course finite and actually defines an integer. For a representation $\rho \otimes \omega_s$ of W_F, we define $cond(\rho \otimes \omega_s) = cond(\rho)$. For a W-D representation $\rho' = (\rho, N)$ we define the conductor as

$$c(\rho') = c(\rho) + dimV^I - dim(KerN)^I \tag{4.0.5}$$

so that it equals $c(\rho)$ when $N = 0$.

The definition of the epsilon factors is more subtle than in the case of $GL(n)$. We will denote by V a finite-dimensional complex representation of W_F, by ψ a non-trivial additive character of F, by dx an additive Haar measure on F.

Recall that, for $GL(1)$ over a global field, we have a *global* functional equation for Hecke L-functions:

$$L(s, \chi) = \epsilon(s, \chi)L(1 - s, \chi^{-1}) \tag{4.0.6}$$

with $L(s, \chi) = \prod_v L(s, \chi_v)$ (product over all the places) and $\epsilon(s, \chi) = \prod_v \epsilon(s, \chi_v, \psi_v)$ independent of the choice of the local ψ_v. Here χ is a Hecke character of a global field (we will take a function field to avoid to consider the infinite primes).The local ϵ factors enter in the local functional equations:

$$\frac{Z(s, \chi_v, \Phi)}{L(s, \chi_v)} = \frac{\epsilon(s, \chi_v, \psi_v)Z(1 - s, \chi_v^{-1}, \hat{\Phi})}{L(1 - s, \chi_v^{-1})} \tag{4.0.7}$$

as was explained in the previous lecture. We show how to deduce 4.0.6 from 4.0.7: first of all we have:

$$L^S(s, \chi) = (\prod_{v \in S} \gamma(s, \chi_v, \psi_v))L^S(1 - s, \chi_v^{-1}) \tag{4.0.8}$$

where S is the set of places containing the archimedean ones and the ramifications places of χ or ψ; L^S denotes the prime-to-S Euler product. Recall that

$$\gamma(s, \chi_v, \psi_v) = \epsilon(s, \chi_v, \psi_v)L(1 - s, \chi_v^{-1})L(s, \chi_v)^{-1}$$

If $v \notin S$, then $\epsilon(s, \chi_v, \psi_v) = 1$.

Then we multiply by $L_S(s, \chi_v, \psi_v)$ and find immediately 4.0.6.

One can prove in general a *global* functional equation:

$$L(s, \rho) = \epsilon(s, \rho)L(1 - s, \hat{\rho}) \tag{4.0.9}$$

and the question is whether one can factor $\epsilon(s, \rho)$ as before. The answer is affirmative and the following theorem defines the local epsilon factors (at $s = 0$, for general s one can twist by a power of the norm character); V denotes a representation of W_F:

Theorem 4.0.22. *There is a unique map $(V, \psi, dx) \mapsto \epsilon(V, \psi, dx) \in \mathbf{C}^*$ such that:*

1. *if $dimV = 1$, it is the epsilon factor defined by Tate;*

2. *it is multiplicative in exact sequences of representations;*
3. *for a finite separable extension E/F, a representation V_E of W_E, there is a factor $\lambda(E/F, \psi, dx_E, dx_F) \in \mathbf{C}^*$ such that*

$$\epsilon(Ind_E^F V_E, \psi) = \lambda(E/F, \psi, dx_E, dx_F)^{dimV_E} \epsilon(V_E, \psi \circ Tr_{E/F}, dx_E) \tag{4.0.10}$$

The basic idea here is the Brauer's induction theorem: if r is a representation of a finite group G then there exist nilpotent subgroups H_i of G and 1-dimensional representations ψ_i of H_i such that r is an integral linear combination of the $Ind_{H_i}^G \psi_i$.

Throughout, we will consider dx as fixed. We record the formula:

$$\epsilon(s, V, \psi) = \epsilon(0, V, \psi) q^{-s(c(\psi)+c(V)dimV)} \tag{4.0.11}$$

For a W-D representation, we put:

$$\epsilon(0, \rho', \psi) = \epsilon(0, \rho, \psi) det(-\rho(\sigma)|V^I/(KerN)^I) \tag{4.0.12}$$

This is not multiplicative in exact sequences. This looks like an ad hoc definition, we'll see that it is motivated by the fundamental example of the elliptic curves with split multiplicative reduction.

5 The Langlands correspondence

Fix $n \geq 1$.

The Langlands correspondence is a bijection between the isomorphism classes of smooth irreducible supercuspidal representations of $GL(n, F)$ and the isomorphism classes of n-dimensional irreducible representations of W_F. Write $\mathcal{A}(n)$ and $\mathcal{G}(n)$ these two sets.

Theorem 5.0.23. *For any $n \geq 1$ there is a map $\sigma : \mathcal{A}(n) \to \mathcal{G}(n)$ such that:*

$$\sigma(\hat{\pi}) = \widehat{\sigma(\pi)} \tag{5.0.13}$$

$$\epsilon(s, \pi, \pi', \psi) = \epsilon(s, \sigma(\pi) \otimes \sigma(\pi'), \psi) \tag{5.0.14}$$

$$L(s, \pi, \pi') = L(s, \sigma(\pi) \otimes \sigma(\pi')) \tag{5.0.15}$$

for any $\pi' \in \mathcal{A}(m)$. Moreover, the determinant of ρ corresponds to the central character of $\sigma(\rho)$. When $n = 1$, the map σ is the usual Artin map of (local) class field theory.

The proof is due to Harris & Taylor and, later, Henniart, when $char(F) = 0$; to Laumon, Rapoport, Stuhler when $char(F) > 0$.

Remark 5.0.24. Henniart's theorem (see previous lecture) implies the uniqueness of σ.

Remark 5.0.25. The map σ extends (in a unique way) to a bijection of the set of semisimple W-D representations of W_F (not necessarily irreducible), with the set of all the irreducible admissible representations of $GL(n)$.

Theorem 5.0.26. *The map σ preserves the conductors.*

This is theorem 2 in Bushnell, Henniart, Kutzko (Ann.E.N.S.31).

From theorem 2.0.10, if p is odd and $n = 2$, the only irreducible representations of W_F are those induced from a character of a separable quadratic extension of F. In this case the map σ is given by the Weil representation (see §6).

Remark 5.0.27. Proposition 2.0.8 corresponds to the fact that any element of $\mathcal{A}(n)$ is of the form $\pi \otimes \chi$, with π having central character of finite order.

But when $p|n$, there are representations of W_F which are not induced from any proper subgroup (Weil, Inv.Math. 27, Koch, Inv.Math.40). In these cases, no explicit construction of σ is known (as far as I know).

Remark 5.0.28. Suppose ρ, ρ' are irreducible representations of Γ_F such that $\sigma^{-1}(\rho) \simeq \sigma^{-1}(\rho')$. It is known that $L(s, \sigma^{-1}(\rho), \sigma^{-1}(\hat{\rho'}))$ has a pole in $s = 0$ if and only if $\widehat{\sigma^{-1}(\rho)} \simeq \sigma^{-1}(\rho')$. Then $L(s, \rho \otimes \hat{\rho'})$ has a pole at $s = 0$ and this implies $\rho \simeq \rho'$.

Theorem 5.0.29. *Let $char(F) = p$ and $n \geq 2$. Let $\rho : \Gamma_F \to GL_n(\mathbf{C})$ an irreducible representation and $\pi \in \mathcal{A}(n)$. If $\epsilon(s, \rho \otimes \tau, \psi) = \epsilon(s, \pi \otimes \sigma(\tau), \psi)$ for any $\tau \in \mathcal{A}(m), m < n$, then $\pi \simeq \sigma(\rho)$.*

Proof. See Henniart, Inv.Math.113. □

The correspondence stated in theorem 5.0.23 is not rational, due to the unitary normalization of the induction. However, one can choose a different normalization as follows: replace π by $\pi \otimes |det|^{-1/2}$. For example, when $n = 2$ the smooth representation of $GL(2)$ corresponding to the sum of characters $\mu_1 \oplus \mu_2$ is the induced representation realized in the space of $\mathbf{C}$-valued functions such that $f(\begin{pmatrix} a & * \\ 0 & b \end{pmatrix} g) = \mu_1(a)\mu_2(b)|b|^{-1} f(g)$.

5.1 Example: Abelian varieties with good reduction

Let A be an abelian variety over a global field K, v a finite place of K, ℓ a prime different fomr the residue characteristic at v, ρ_ℓ the representation of $Gal(\bar{K}/K)$ on $T_\ell A$.

Theorem 5.1.1. *A has potential good reduction at v if and only if the image by ρ_ℓ of the inertia group I is finite.*

For the proof, see Serre & Tate, Ann.Math.88.

Noe let A be an abelian variety with good reduction over a non archimedean local field K of mixed characteristic, and let $\rho : Gal(\bar{K}/K) \to GL(V_\ell A)$ the representation on the Tate module with $\ell \neq$ residue characteristic of K.

Proposition 5.1.2. *ρ is semisimple.*

Proof. We have to show: $\rho(\sigma)$ is diagonalizable (σ is the Frobenius). If $\bar{A}$ is the special fiber of the Neron model of A, we have $V_\ell A \simeq V_\ell \bar{A}$ as Galois modules. It is well known that $Gal(\bar{k}/k)$ acts semisimply on $V_\ell \bar{A}$ and thus ρ splits as a direct sum of characters.

□

Actually, the same is true in case of potential good reduction.

6 Elliptic curves and special representations

Convention: in the following, the Frobenius automorphism is the geometric one.

We recall some facts on special representations: if μ_1, μ_2 are characters of F^*, then $\pi = Ind_P^{GL(2)}(\mu_1 \otimes \mu_2)$ is reducible if and only if $\mu_1\mu_2^{-1} = |\,|^{\pm 1}$, so we have two cases:

- if $\mu_1\mu_2^{-1} = |\,|$, then we have a non-split exact sequence:

$$0 \to \pi_1 \to \pi \to \pi_2 \to 0$$

  with π_2 one-dimensional and π_1 is the unique irreducible subrepresentation of π. Moreover, the Kirillov model of π_1 consists of the locally constant functions f on F^* that vanish for large x and such that there is a constant C for which $f(x) = C|x|^{1/2}\mu_1(x)$ if $|x|$ is small. In particular, $L(s, \pi_1) = L(s, \mu_1)$. It is customary to denote π_1 by $\sigma(\mu_1, \mu_2)$.

- if $\mu_1\mu_2^{-1} = |\,|^{-1}$, then we have a non-split exact sequence:

$$0 \to \pi_3 \to \pi \to \pi_4 \to 0$$

  with π_3 one-dimensional and π_4 is the unique irreducible quotient of π. Moreover, the Kirillov model of π_4 consists of the locally constant functions f on F^* that vanish for large x and such that there is a constant C for which $f(x) = C|x|^{1/2}\mu_2(x)$ if $|x|$ is small. In particular, $L(s, \pi_4) = L(s, \mu_2)$. It is customary to denote π_4 by $\sigma(\mu_1, \mu_2)$.

Consider the elliptic curve over $\mathbf{Q}$: $E = \Gamma_0(11)\backslash\mathcal{H}$. We want to write down the smooth representations corresponding to the local Galois representations on the torsion points of E at all the finite places. There are two cases to consider. The curve E has split multiplicative reduction at 11. Let $\ell \neq p$ be rational prime.

6.1 Bad reduction

Fix $p = 11$; then we have an exact sequence of $G_{\mathbf{Q}_p}$-modules:

$$1 \to T_\ell\mu \to T_\ell E \to \mathbf{Z}_\ell \to 0 \tag{6.1.1}$$

where $\mu = \varprojlim \mu_{\ell^n}$, the projective limit of the groups of roots of unity of ℓ-power order, so that the Galois representation $\rho : G_{\mathbf{Q_p}} \to Aut(V_\ell E)$ has the form:

$$\rho = \begin{pmatrix} \chi & c \\ & 1 \end{pmatrix} \tag{6.1.2}$$

where χ is the p-adic cyclotomic character. We get from this a W-D representation (ρ', N), as in theorem 3.0.16. Note that it is reducible, so we cannot take $N = 0$. A look at conductors shows that $\sigma^{-1}(\rho)$ must be a special representation.

Since $p = 11$, ρ is indecomposable, $N \neq 0$ and we claim that ρ' has conductor p, so that $\sigma^{-1}(\rho)$ is a special representation. More precisely, we have $N = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. This is the special representation of W_F of dimension 2. Let us compute the conductor of ρ: from the definition we see that $c(\rho') = c(\rho)$ in this case and $c(\rho) = 1$ because the wild inertia acts trivially.

We know that $\sigma(\rho)$ is a quotient or a subrepresentation of an induced: $\pi =_U Ind_P^G(\mu_1 \otimes \mu_2)$, for some characters μ_1, μ_2 of $\mathbf{Q}_p^*$. The central character of $\sigma^{-1}(\rho)$ must correspond to χ^{-1}, so it is $||$. Therefore:

$$\mu_1\mu_2 = |\,| \tag{6.1.3}$$

Moreover, we know that $\mu_1\mu_2^{-1} = |\,|^{\pm 1}$ because π is reducible, thus we have either

$$\mu_1 = 1, \quad \mu_2 = |\,| \tag{6.1.4}$$

or

$$\mu_1 = |\,|, \quad \mu_2 = 1 \tag{6.1.5}$$

Note that $\sigma(1, |\,|) \simeq \sigma(|\,|, 1)$ (by comparison of Kirillov models, for example). We know that $L(s, \sigma(1, |\,|)) = L(s, \mu_2) = (1 - |p|q^{-s})^{-1}$.

Now,

$$L(s, \rho) = det(1 - \rho(Frob)q^{-s}|Ker(N)^I)^{-1} = (1 - \chi(Frob)q^{-s})^{-1} \tag{6.1.6}$$

so that

$$L(s, \rho) = L(s, \sigma(1, |\,|)) \tag{6.1.7}$$

One can also check that

$$L(s,\rho^\vee) = L(s,\widehat{\sigma(1,||)}) \quad (6.1.8)$$

Moreover, the epsilon factor of $\sigma(1,||)$ is:

$$\epsilon(s,\sigma(1,||),\psi) = \epsilon(s,1,\psi)\epsilon(s,||,\psi)L(1-s,||^{-1})L(s,1)^{-1} = -q^{-s} \quad (6.1.9)$$

if ψ is unramified and this equals the epsilon factor of ρ.

6.2 Good reduction

If $p \neq 11$, it is a place of good reduction, and we know that $N = 0$ (Neron-Ogg-Shafarevich) in this case. Moreover, $\sigma^{-1}(\rho)$ is a principal series, induced by two unramified characters, determined by the characteristic polynomial of ρ. We must have:

$$\mu_1\mu_2(p) = p^{-1} \quad (6.2.1)$$

and

$$L(s,\pi) = L(s,\rho) \quad (6.2.2)$$

We see immediately that:

$$L(s,\rho) = det(1-\rho(Frob)p^{-s}|V)^{-1} \quad (6.2.3)$$

where $V = V_\ell E$. Moreover, the characteristic polynomial of $\rho(Frob)$ is determined by:

$$det(\rho(Frob)) = p^{-1} \quad (6.2.4)$$

$$tr\rho(Frob) = a_p/p \quad (6.2.5)$$

where $a_p = p + 1 - |E(\mathbf{F}_p)|$.

In order to compute $L(s,\pi)$, we recall the formula for the unramified Whittaker function W: we have

$$W(\begin{pmatrix} p^m & \\ & 1 \end{pmatrix}) = (1-p^{-1}\alpha_1\alpha_2^{-1})p^{-m/2}\frac{\alpha_1^{m+1}-\alpha_2^{m+1}}{\alpha_1-\alpha_2} \quad (6.2.6)$$

if $m \geq 0$, where $\alpha_i = \mu_i(p)$. If $m < 0$, W is 0.

Then we can compute:

$$L(s,\pi) = (1-\mu_1(p)p^{-s})^{-1}(1-\mu_2(p)p^{-s})^{-1} \quad (6.2.7)$$

Therefore we find:

$$\mu_1(p)\mu_2(p) = p^{-1} \quad (6.2.8)$$

and

$$\mu_1(p) + \mu_2(p) = a_p p^{-1} \tag{6.2.9}$$

(note that exchanging μ_1 and μ_2 doesn't change the isomorphism class of π).

6.3 Example: some supercuspidal representations

Theorem 6.3.1. *Let τ be a smooth irreducible representation of $GL_2(O_F)Z$, with no vectors fixed by $U(O_F)$. Then $Ind^G_{GL_2(O_F)Z}\tau$ is supercuspidal.*

Proof. We will prove only the first claim of the theorem.

- admissibility: we have to that the finite-dimensionality of the space of functions f on G with values in the space of τ, such that $f(\gamma g k) = \tau(\gamma) f(g)$ for any $\gamma \in GL_2(O_F)Z$, $k \in GL_2(\mathfrak{p}^m)$. Put $a = \begin{pmatrix} 1 & \\ & \varpi^m \end{pmatrix}$, $u \in U(O_F)$. Then

$$\tau(k^{-1}uk) f(k^{-1}ak) = f(k^{-1}uak) = f(k^{-1}akk^{-1}a^{-1}uak) = f(k^{-1}ak)$$

- supercuspidality: we prove that $Hom_U(Ind(\tau), \mathbf{C}) = 0$. Let $\phi : Ind(\tau) \to \mathbf{C}$. There is a function $F : G \to \hat{\tau}$ such that F is U-invariant, $F \in Ind(\hat{\tau})$, $\phi(f) = \int_{G/Z} < F(g), f(g) > dg$ for all $f \in Ind(\tau)$.

  With a as before: $\hat{\tau}(u)F(a) = F(ua) = F(a(a^{-1}ua)) = F(a)$ for any $u \in U$. Therefore $F(a) = 0$ and $F = 0$ by the Cartan decomposition.

□

7 Functoriality

If $r : GL_n(\mathbf{C}) \to GL_m(\mathbf{C})$ is a homomorphism, then we have a map $\mathcal{G}(n) \to \mathcal{G}(m)$ and thus, from 5.0.23, a map: $\mathcal{A}(n) \to \mathcal{A}(m)$.

A trivial example of this is when $r = det$, $m = 1$, which corresponds to taking the central character of a representation in $\mathcal{A}(n)$.

If E/F is a finite separable extension of degree d, then we must have maps: $\mathcal{A}_E(n) \to \mathcal{A}_F(dn)$ (induction) and $\mathcal{A}_F(n) \to \mathcal{A}_E(n)$ (base-change). For a short presentation of base-change, see my talk at the Intercity Seminar.

8 Weil representations

This section is intended to be an ultra-short introduction on Weil representations. For a serious introduction to this topic, we suggest D.Prasad 'Weil representation, Howe duality, and the Theta correspondence'.

Assume $char(F) \neq 2$. Let W be a 2m-dimensional F-vector space with a non degenerate symplectic form $<,>: W \times W \rightarrow F$. Define the Heisenberg group:

$$H(W) = \{(w,t) \in W \times F\} \tag{8.0.1}$$

with composition law:

$$(w,t)(w',t') = (w + w', t + t' + < w, w' > /2) \tag{8.0.2}$$

It fits in a central exact sequence:

$$0 \rightarrow F \rightarrow H(W) \rightarrow W \rightarrow 0 \tag{8.0.3}$$

Theorem 8.0.2. *For any non-trivial character $\psi : F \rightarrow \mathbf{C}^*$, there exists a unique isomorphism class of irreducible smooth representations $\rho_\psi : H(W) \rightarrow GL(V)$ on which F acts via ψ.*

This is a very general theorem due to Stone and von Neumann.

The symplectic group $Sp(W)$ acts on $H(W)$ by $g(w,t) = (gw,t)$, therefore there exists a map $g \mapsto \omega_\psi(g)$ from $H(W)$ to $PGL(V)$ such that

$$\rho_\psi(gw,t) = \omega_\psi(g)\rho_\psi(w,t)\omega_\psi(g)^{-1} \tag{8.0.4}$$

It is a crucial fact that, since W has even dimension, the map ω_ψ is actually a 'true' representation of $Sp(W)$. We will not prove this fact, we will content ourselves to exhibit the resulting representation. Several models of this representation are known, we describe the Schrodinger model: let $W = X \oplus Y$ be a complete polarization, then we let $H(W)$ act on $C_c^\infty(X)$ as follows:

$$\rho_\psi(w_1)f(x) = f(x + w_1) \tag{8.0.5}$$

$$\rho_\psi(w_2)f(x) = \psi(< x, w_2 >)f(x) \tag{8.0.6}$$

$$\rho_\psi(t)f(x) = \psi(t)f(x) \tag{8.0.7}$$

where $t \in F$, $x, w_1 \in W_1, w_2 \in W_2$.

Then let $Sp(W)$ act on $C_c^\infty(X)$ as follows:

$$\begin{pmatrix} A & \\ & {}^tA^{-1} \end{pmatrix} f(x) = |det(A)|^{1/2} f({}^tAx) \tag{8.0.8}$$

$$\begin{pmatrix} 1 & B \\ & 1 \end{pmatrix} f(x) = \psi({}^txBx/2)f(x) \tag{8.0.9}$$

$$\begin{pmatrix} & 1 \\ -1 & \end{pmatrix} f(x) = \gamma \hat{f}(x) \tag{8.0.10}$$

where $\gamma^8 = 1$ (for the precise value, see Weil's original paper 'Sur certains groupes d'operateurs unitaires', Acta Mathematica 111). It is not hard to check that the action just defined satisfies 8.0.4.

If $dim W = 2$, then $Sp(W) = SL_2(F)$ and $X = F$. Let K be a separable quadratic extension of F. One can extend this representation to a representation of $SL_2(F)$ on $C_c^\infty(K)$: just replace the argument of ψ in 8.0.9 by its norm. It extends even to a representation of $GL_2(F)_+$ (matrices with determinant in $N(K^*)$): let Ω be a finite-dimensional irreducible representation of K^* on a complex vector space U. Then $SL_2(F)$ acts on $C_c^\infty(K) \otimes_{\mathbf{C}} U$ with trivial action of $SL_2(F)$ on U and this action verifies:

$$\begin{pmatrix} A & \\ & 1 \end{pmatrix} f(x) = |det(A)|^{1/2} \Omega(h) f(xh) \tag{8.0.11}$$

if $A = N(h)$ is in the image of the norm $K^* \to F^*$. The groups $GL_2(F)_+$ has index 2 in $GL_2(F)$. We extend further this representation to a representation of $GL_2(F)$, by induction. One can prove that the isomorphism class of the resulting representation doesn't depend on the choice of ψ.

Moreover, this representation, say π_Ω, is supercuspidal (and irreducible) if and only if Ω doesn't factor through the norm and we have:

$$L(s, \Omega) = L(s, \pi_\Omega) \tag{8.0.12}$$

$$\pi_{\hat{\Omega}} = \widehat{\pi_\Omega} \tag{8.0.13}$$

Remark 8.0.3. This theory can be seen as a representation-theoretic view on theta functions. If X is a complex abelian variety and L a line bundle on it, one defines a Heisenberg group $G(L)$ as the group of automorphisms of L induced by translations on X. Then $\mathbf{C}^*$ injects in the center of $G(L)$. This Heisenberg group acts on $\Gamma(X, L)$ by:

$$(\phi, \Phi) f = \Phi \circ f \circ \phi^{-1} \tag{8.0.14}$$

Assume for example the genus of X to be 1. If $\phi(z) = z - a, a \in \mathbf{C}$, and $\Phi(w, z) = (w exp(\pi i Im(a)(2z - a)), z - a)$ and f is identified with a complex function on $\mathbf{C}$, then (ϕ, Φ) acts on f by

$$f \mapsto f(z + a) exp(\pi i Im(a)(2z + a)) \tag{8.0.15}$$

which is the usual action on theta functions.