L-functions for GL(n)

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1 Introduction

In this lecture, we introduce the L-functions and the epsilon factors for $GL(n)$. We consider only generic representations; this suffices for the application to the Langlands correspondence, because supercuspidals are generic and they correspond to the irreducible Galois representations.

We warn the reader that this is not true for other groups: Piatetski-Shapiro has found examples of supercuspidal representations of GSp_4 which are not generic. However, work of Gelbart, Piatetski-Shapiro and Rallis provides a large supply of L-functions for smooth representations of classical groups (Lecture Notes 1254). Their work extends the approach of Godement, Jacquet. We have chosen to work with Whittaker models because they are rather explicit and we are particularly interested in the characterization of supercuspidals in terms of epsilon factors (Theorem 3.0.7).

2 Zeta integrals

Let π be a smooth irreducible representation of $GL(n)$, $n \geq 2$, and π' a smooth irreducible representation of GL_m, $m < n$. We assume that both are generic. Then, if W, W' are functions in the respective Whittaker models, we define:

$$Z(s, W, W') = \int_{U_m \backslash GL_m} W\begin{pmatrix} h & \\ & I_{n-m} \end{pmatrix} W'(h) |det(h)|^{s-(n-m)/2} dh \qquad (2.0.1)$$

where U_m is the unipotent radical of the Borel subgroup of $GL(m)$.

When $m = 1$, we replace W' by π'. For $n = 1$, the theory of zeta-integrals is the subject of Tate's thesis; for a Schwartz function ϕ on F^* and a quasicharacter χ, one defines:

$$\int_{F^*} \chi(x)\phi(x)|x|^{s-1} dx \qquad (2.0.2)$$

When $m = n$, we define:

$$Z(s,W,W',\Phi)=\int_{U\backslash G} W(g)W'(g)\Phi(e_n g)|det(g)|^s dg \tag{2.0.3}$$

where Φ is a Schwartz function on F^n.

Remark 2.0.1. When $n=2$, $m=1$, the definition is simply the local analogue of the Mellin transform. For general m,n, it is the local analogue of the global integral:

$$\int_{GL_m(F)\backslash GL_m(\mathbf{A})} \mathcal{P}\phi(\begin{pmatrix} h & \\ & 1\end{pmatrix})\phi'(h)|det(h)|^{s-1/2}dh \tag{2.0.4}$$

where ϕ,ϕ' are automorphic forms on GL_n, GL_m respectively and

$$\mathcal{P}\phi(\begin{pmatrix} h & \\ & 1\end{pmatrix}) = |det(h)|^{-(n-m-1)/2}\sum_{\gamma\in U_m(F)\backslash GL_m(F)} W'((\begin{pmatrix} \gamma & \\ & 1\end{pmatrix})(\begin{pmatrix} h & \\ & 1\end{pmatrix}) \tag{2.0.5}$$

where W' on the right is a global Whittaker function.

Theorem 2.0.2. *1. the integrals 2.0.1 and 2.0.3 converge for $re(s) >> 0$.*

2. 2.0.1, 2.0.3 define rational functions in q^{-s} and hence extend meromorphically to $\mathbf{C}$.

3. there is a polynomial $P(X)$ in $\mathbf{C}[X]$, with constant term =1, such that $Z(s,W,W')P(q^{-s})$ is an entire function for any W,W' and such that $Z(s,W,W')=P(q^{-s})^{-1}$ for an appropriate choice of W,W'. The same is true for $Z(s,W,W',\Phi)$.

We define:

$$L(s,\pi,\pi')=P(q^{-s})^{-1} \tag{2.0.6}$$

with P of minimal degree. Of course, $L(s,\pi,\pi')$ is uniquely determined.

The theorem is deduced from:

Theorem 2.0.3. *Let A be the maximal torus in G. There are finitely many finite functions x_i on A such that, for any Whittaker function W there are Schwartz-Bruhat functions ϕ_j on F^{n-1} with:*

$$W(a)=\sum x_i(a)\phi_i(\alpha_1(a),...,\alpha_{n-1}(a)) \tag{2.0.7}$$

where $a=diag(a_1,...,a_{n-1},1)$, $\alpha_i(a)=a_i/a_{i+1}$.

A finite function is a continuous function whose translates span a finite-dimensional vector space.

This theorem (and the previous one) is due to Jacquet, Piatetski-Shapiro, Shalika (Am.J.Math. 105). In order to apply it, one has to use the Iwasawa decomposition:

$$GL_n(F) = GL_n(O_F)AGL_n(O_F) \tag{2.0.8}$$

and the fact that the $GL_n(O_F)$-translates of W, W' span finite-dimensional spaces.

Here are some explicit formulae for $L(s, \pi, \pi')$:

Theorem 2.0.4. *If π, π' are unramified, we can write $\pi = Ind(\chi_1 \otimes ... \otimes \chi_n)$, $\pi' = Ind(\mu_1 \otimes ... \otimes \mu_m)$. Then:*

$$L(s, \pi, \pi') = \prod_{i=1}^{n}\prod_{j=1}^{m}(1 - \chi_i(\varpi)\mu_j(\varpi))^{-1} \tag{2.0.9}$$

If π, π' are supercuspidal and $n > m$, then:

$$L(s, \pi, \pi') = 1 \tag{2.0.10}$$

If $n = m$ we have:

$$L(s, \pi, \pi') = \prod(1 - \alpha q^{-s})^{-1} \tag{2.0.11}$$

with the product over all $\alpha = q^{s_0}$ such that $\hat{\pi} \simeq \pi' \otimes |det|^{s_0}$.
Examples:

- take $n = 2$, π unramified and $\pi' = \chi$ an unramified character. Then $\pi =_U Ind_P^G(\chi_1 \otimes \chi_2)$. One can compute the Whittaker function W associated to the normalized spherical vector for π (i.e. the $GL_2(O_F)$-invariant function f with $f(1) = 1$): first of all

  $$W(\begin{pmatrix} \varpi^m & \\ & 1 \end{pmatrix}) = W(\begin{pmatrix} \varpi^m & \\ & 1 \end{pmatrix}\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix}) = \psi(\varpi^m x)W(\begin{pmatrix} \varpi^m & \\ & 1 \end{pmatrix})$$

  for any $x \in O_F$. If we assume $cond(\psi) = O_F$, then $W(\begin{matrix} \varpi^m & \\ & 1 \end{matrix}) = 0$ for $m < 0$.

  For $m \geq 0$, one can prove:

  $$W(\begin{matrix} \varpi^m & \\ & 1 \end{matrix}) = (1 - q^{-1}\chi_1\chi_2^{-1}(\varpi))q^{-m/2}(\chi_1(\varpi)^{m+1} - \chi_2(\varpi)^{m+1})/(\chi_1(\varpi) - \chi_2(\varpi)) \tag{2.0.12}$$

  In general, assuming π irreducible and $\chi_1 \neq \chi_2$, one can prove that the Kirillov model of π consists of locally constant functions f on F^* such that there are constants C_1, C_2 with:

  $$f(x) = C_1|x|^{1/2}\chi_1(x) + C_2|x|^{1/2}\chi_2(x) \tag{2.0.13}$$

for x small and $f(x) = 0$ for x large.

From this, it is easy to compute $L(s, \pi, \chi) = Z(s, W, \chi)$.

The formula for spherical Whittaker functions for general n can be found in Casselman, Shalika (Comp.Math.41). In general, one can prove:

$$L(s, Ind(\rho \otimes \rho')) = L(s, \rho)L(s, \rho') \tag{2.0.14}$$

for any ρ, ρ'.

The fact that $L(s, \pi) = 1$ for supercuspidal π follows from the fact that the Kirillov model for π is precisely the space of locally constant, compactly supported functions on F^*, so that the zeta-integral is automatically a polynomial in $\mathbf{C}[q^s, q^{-s}]$. We recall some properties of Kirillov models in an appendix.

- Choose an unramified character χ of F^*; we form the induced representation: $\pi =_U Ind_P^G(\chi_1 \otimes \chi_2 \otimes ... \otimes \chi_n)$, where $\chi_n = \chi$, $\chi_j = \chi_{j+1}| |^{-1}$. This representation is reducible and, from the Bernstein-Zelevinsky classification, we know that it has a unique irreducible (ramified) quotient and a unique irreducible (unramified) subrepresentation. The L-function of the subrepresentation is $L(s, \chi)$ and the L-function of the quotient is 1. Again, this follows from the form of the Kirillov model of π; when $n = 2$, it consists of locally constant functions f on F^* such that there is a constant C with:

$$f(x) = C|x|^{1/2}\chi(x)$$

for x small and $f(x) = 0$ for large x.

Proposition 2.0.5. *Let τ be a smooth irreducible generic representation of GL_m, $m < n$. Then, if π is a supercuspidal representation of GL_n:*

$$L(s, \pi, \tau) = 1 \tag{2.0.15}$$

This is lemma 3.3 of Henniart (Inv.Math. 113). The same paper contains general formulae for $L(s, \pi)$, according to the Bernstein-Zelevinsky classification.

Now we turn to the functional equation: define

$$\tilde{W}(g) = W(w^t g^{-1}) \tag{2.0.16}$$

for any Whittaker function W on $GL(n)$.

Theorem 2.0.6.

- *$n > m$. There is a unique function $\epsilon(s, \pi, \pi', \psi)$ such that:*

$$\frac{Z(s,W,W')}{L(s,\pi,\pi')}\epsilon(s,\pi,\pi',\psi) = \frac{Z(1-s,\tilde{W},\tilde{W'})}{L(1-s,\hat{\pi},\hat{\pi}')} \tag{2.0.17}$$

- *$n = m$. There is a unique function $\epsilon(s,\pi,\pi',\psi)$ such that:*

$$\frac{Z(s,W,W',\Phi)}{L(s,\pi,\pi')}\epsilon(s,\pi,\pi',\psi) = \frac{Z(1-s,\tilde{W},\tilde{W'},\hat{\Phi})}{L(1-s,\hat{\pi},\hat{\pi'})} \tag{2.0.18}$$

where $\hat{\Phi}$ is the Fourier transform of Φ.

Moreover, $\epsilon(s,\pi,\pi',\psi)$ has the form cq^{-fs}, where f is a non-negative integer.

Note that ϵ is independent of W,W'. It is clear from the definition that $\epsilon(s,\pi,\pi',\psi)$ is a polynomial in q^s, q^{-s}. On the other hand, we have:

$$\epsilon(s,\pi,\pi',\psi)\epsilon(1-s,\hat{\pi},\hat{\pi'},\bar{\psi}) = \omega(-1) \tag{2.0.19}$$

and this implies that $\epsilon(s,\pi,\pi',\psi)$ is in fact a monomial in q^{-s}.

If we change character, it is easy to see how the epsilon factor changes. For example, if $\psi'(x) = \psi(ax)$ for a fixed $a \neq 0$ and $m = 1$, $\pi' = 1$, then:

$$\epsilon(s,\pi,1,\psi') = \epsilon(s,\pi,1,\psi)\omega(a)|a|^{n(s-1/2)} \tag{2.0.20}$$

where ω is the central character of π.

If π,π',ψ are unramified, we can choose W,W' such that the zeta integrals coincide with the L-functions and thus $\epsilon(s,\pi,\pi') = 1$ (one can also prove $f = 0$).

Example. In Tate's thesis, section 2.5, one can find the computation of the epsilon factors or characters:

- $$\epsilon(s,\chi,\psi) = \chi(\varpi)^m q^{(1/2-s)m} \tag{2.0.21}$$

  if m is the exponent of the conductor of ψ and χ is unramified;

- In general:

  $$\epsilon(s,\chi,\psi) = \omega(\varpi^{m+c})q^{(1/2-s)(m+c)}g(\chi,\psi) \tag{2.0.22}$$

  if χ is ramified with conductor $\mathfrak{p}^c$ and

  $$g(\chi,\psi) = Vol(O_F^*)^{-1}\int_{O^*}\chi^{-1}\psi(\varpi^{-m-c}x)dx \tag{2.0.23}$$

  where dx is now the self-dual measure w.r.t ψ.

It is also useful to consider the following local factor:

$$\gamma(s,\pi,\pi',\psi) = \epsilon(s,\pi,\pi',\psi)\frac{L(1-s,\hat{\pi},\hat{\pi}')}{L(s,\pi,\pi')} \tag{2.0.24}$$

3 Henniart's theorem

We have the following beautiful theorem (Henniart, Inv.Math.113)

Theorem 3.0.7. *a)Let $n \geq 2$, π_1, π_2 two smooth irreducible generic representations. Assume that:*

$$\gamma(s, \pi_1, \pi', \psi) = \gamma(s, \pi_2, \pi', \psi) \tag{3.0.25}$$

for any smooth irreducible generic representation π' of $GL(n-1)$. Then $\pi_1 \simeq \pi_2$.

b)Let $n \geq 2$, π_1, π_2 two supercuspidal representations of $GL(n)$. Assume that, for any $m < n$, and for any supercuspidal representation π' of $GL(m)$, we have:

$$\epsilon(s, \pi_1, \pi', \psi) = \epsilon(s, \pi_2, \pi', \psi) \tag{3.0.26}$$

Then $\pi_1 \simeq \pi_2$.

This theorem implies the uniqueness of the local Langlands correspondence (Henniart, loc.cit).

Proof. (sketch).We sketch the proof of part a). We define

$$\mathcal{S} = \{(W_1, W_2) \in \mathcal{W}_{\pi_1,\psi} \oplus \mathcal{W}_{\pi_2,\psi}; W_1|_{GL_{n-1}} = W_2|_{GL_{n-1}}\} \tag{3.0.27}$$

We want to prove that $\mathcal{S}$ is stable for $GL(n)$, $\mathcal{S} \neq \mathcal{W}_{\pi_1,\psi} \oplus \mathcal{W}_{\pi_2,\psi}$ and $\mathcal{S}$ projects non-trivially on $\mathcal{W}_{\pi_1,\psi}$ and $\mathcal{W}_{\pi_2,\psi}$. Then, by irreducibility of π_1, π_2, $\mathcal{S}$ is the graph of an isomorphism $\pi_1 \simeq \pi_2$.

To begin with, we prove that $\mathcal{S}$ is G-stable. Define $\tilde{\mathcal{S}}$ by analogy with $\mathcal{S}$, taking duals.

The subgroup P_n introduced in the previous lecture stabilizes $\mathcal{S}$ and $\tilde{\mathcal{S}}$. From lemma 3.0.8, if $g \in^t P_n$ then: $g(W_1, W_2) \in \mathcal{S}$ if and only if $(\widetilde{gW}_1, \widetilde{gW}_2) \in \tilde{\mathcal{S}}$.

Now:

$$\widetilde{gW}_i(h) = W_i(w^t h^{-1} g) = \tilde{W}_i(h^t g^{-1}) =^t g^{-1} \tilde{W}_i(h)$$

shows, with the next lemma, that ${}^tP_n$ stabilizes $\mathcal{S}$. But P_n and ${}^tP_n$ generate $GL_n(F)$.

It remains to prove that $\mathcal{S} \neq \mathcal{W}_{\pi_1,\psi} \oplus \mathcal{W}_{\pi_2,\psi}$ and $\mathcal{S}$ projects non-trivially on $\mathcal{W}_{\pi_1,\psi}$ and $\mathcal{W}_{\pi_2,\psi}$. This follows from lemma 4.0.9. $\square$

Lemma 3.0.8. *We have:* $(W_1, W_2) \in \mathcal{S}$ *if and only if* $(\tilde{W}_1, \tilde{W}_2) \in \tilde{\mathcal{S}}$.

Proof. We have the equalities:

$$Z(s, W_1, W') = Z(s, W_2, W') \tag{3.0.28}$$

if $(W_1, W_2) \in \mathcal{S}$. Similarly:

$$Z(s, \tilde{W}_1, \tilde{W}') = Z(s, \tilde{W}_2, \tilde{W}') \tag{3.0.29}$$

Consider the function H_r on $GL(n-1)$ such that $H_r(g) = 0$ if $|det(g)| \neq q^r$ and $H_r(g) = \tilde{W}_1(\begin{smallmatrix} g & \\ & 1 \end{smallmatrix}) - \tilde{W}_2(\begin{smallmatrix} g & \\ & 1 \end{smallmatrix})$ if $|det(g)| = q^r$. Then 3.0.28 implies:

$$\int_{U_{n-1}\backslash GL_{n-1}} H_r(g)\tilde{W}'(g)dg = 0 \tag{3.0.30}$$

for any integer r. This implies $H_r = 0$ (Jacquet, Piatetski-Shapiro, Shalika, Math.Ann. 256, lemma 3.5) and thus $(\tilde{W}_1, \tilde{W}_2) \in \tilde{\mathcal{S}}$. The other implication follows from the functional equation, together with the hypothesis on the epsilon factors. □

Part b) of theorem 3.0.7 follows immediately from a) and the proposition 2.0.5.

4 Three lemmas on Whittaker functions

Lemma 4.0.9. *Let π be a smooth irreducible generic representation of G and H a smooth function on $GL_{n-1}(F)$, with compact support modulo U_{n-1} and verifying:*

$$H(ug) = \psi(u)H(g) \tag{4.0.31}$$

Then there is a function $W \in \mathcal{W}_{\pi,\psi}$ such that:

$$W(\begin{pmatrix} g & \\ & 1 \end{pmatrix}) = H(g) \tag{4.0.32}$$

The proof is in: Bernstein,Zelevinsky, Russ.Math.Surveys 31, §5.

Lemma 4.0.10. *Let H be a smooth function on $GL(n)$ such that:*

$$H(ug) = \bar{\psi}(u)H(g) \tag{4.0.33}$$

and H is compactly supported modulo U. If

$$\int_{U\backslash G} H(g)W(g)dg = 0 \tag{4.0.34}$$

for any Whittaker function $W \in \mathcal{W}_{\tau,\psi}$ and for any irreducible generic τ, then $H = 0$.

Lemma 4.0.11. *Let π be an irreducible generic representations of G and $W \in \mathcal{W}_{\pi,\psi}$. For any m there is a compact subset C_m of G such that if $|det(g)| = q^m$ and $W\begin{pmatrix} g & \\ & 1 \end{pmatrix} \neq 0$, then $g \in UC_m$.*

5 Appendix: Kirillov models

We discuss the theory for $GL(2)$ (the theory for $GL(n)$ is essentially the same). Then $\mathcal{K}$ is a space of functions on F^* such that:

$$\pi \begin{pmatrix} a & \\ & 1 \end{pmatrix} f(x) = f(ax) \tag{5.0.35}$$

$$\pi \begin{pmatrix} 1 & a \\ & 1 \end{pmatrix} f(x) = \psi(ax) f(x) \tag{5.0.36}$$

Since π is smooth, f is stabilized by an open subgroup of F^*, therefore it is locally constant. On the other hand f is stabilized by the matrices $\begin{pmatrix} 1 & a \\ & 1 \end{pmatrix}$ with a small enough, therefore $f(x) = \psi(ax)f(x)$ if a is small enough; this implies that $f(x) = 0$ for $|x|$ large.

Moreover,

$$\pi \begin{pmatrix} 1 & a \\ & 1 \end{pmatrix} f(x) - f(x) = (\psi(ax) - 1) f(x)$$

implies $\pi \begin{pmatrix} 1 & a \\ & 1 \end{pmatrix} f(x) = f(x)$ for $|x|$ small.

Therefore, if π is supercuspidal, its space V is contained in the space of locally constant and compactly supported functions on F^*. On the other hand, the last space is irreducible under the action of P_2. In fact, if f is in an invariant subspace of $C_c^\infty(F^*)$, then we consider:

$$f'(a) = \int_F \phi(x) \pi(\begin{matrix} 1 & x \\ & 1 \end{matrix}) f(a) dx \tag{5.0.37}$$

for $\phi \in C_c^\infty(F^*)$. Then $f'(a) = \hat{\phi}(a) f(a)$ and we can choose ϕ such that $\hat{\phi}(a) f(a)$ is equal to the characteristic function of a neighborhood of a where f is constant.