Whittaker models for GL(n)

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In this lecture, we will introduce the Whittaker models for $GL(n)$, which we will use in the next lecture to define the L-functions and epsilon factors of smooth representations.

Let F be a local non-archimedean field as usual, $\pi : G = GL(n, F) \to GL(V)$ a smooth irreducible representation. We fix a non trivial additive character ψ of F. Recall that all the other characters are obtained by multiplication: $x \mapsto \psi(ax)$, for some $a \in F$.

We extend ψ to a character of the unipotent radical as follows:

$$\psi(u) = \psi(\sum_{i=1}^{n-1} u_{i,i+1}) \tag{0.0.0.1}$$

Definition 0.0.0.1. A Whittaker functional on V is a linear functional $\Lambda : V \to \mathbf{C}$ such that:

$$\Lambda(\pi(u)v) = \psi(u)\Lambda(v) \tag{0.0.0.2}$$

for any $u \in U$, $v \in V$.

We say π is generic if it admits non-zero Whittaker functionals.

We have the following theorem, by Gelfand and Kazhdan:

Theorem 0.0.0.2.

1. *if π is supercuspidal and infinite-dimensional, it is generic.*
2. *Whittaker functionals are unique up to scalars.*
3. *π is generic if and only if its contragredient is generic.*
4. *if π is generic and unramified, then $\pi \simeq Ind_P^G(\chi_1 \otimes ... \otimes \chi_n)$, where P is the Borel and χ_i are unramified characters.*
5. *if $n = 2$, every irreducible, infinite-dimensional π is generic.*
6. *the above statements do not depend on the choice of $\psi \neq 1$.*

We will only sketch the proof, for details, see Bernstein & Zelevinsky, Russ.Math.Surveys 31.

First, we define the Whittaker models: consider the space:

$$\mathcal{W}_{\pi,\psi} = \{W_v : G \to \mathbf{C}; W_v(g) = \Lambda(\pi(g)v), v \in V\} \tag{0.0.0.3}$$

for a fixed Whittaker functional $\Lambda \neq 0$ (taking a different Λ, we get an isomorphic space, because of point 2 of the theorem).

We let G act on this space by right translation, then: $gW_v = W_{gv}$. Then $\mathcal{W}_{\pi,\psi}$ is irreducible and the map $v \mapsto W_v$ is a G-isomorphism. We call 0.0.0.3 the Whittaker model for π (it doesn't depend on ψ, up to isomorphisms).

An example of Whittaker functional: take $n = 2$, $\pi = Ind_P^G(\chi_1 \otimes \chi_2)$ and, assuming convergence, define Λ by:

$$f \mapsto \int_F f\begin{pmatrix} 1 & \\ x & 1 \end{pmatrix}\psi(-x)dx \tag{0.0.0.4}$$

Gelfand and Kazhdan proved the following theorem (B-Z, §7):

Theorem 0.0.0.3. *If π is irreducible, $\hat{\pi} \simeq \tilde{\pi}$, where $\tilde{\pi}(g) = \pi(w^t g^{-1} w^{-1})$ and*

$$w = \begin{pmatrix} & & & 1 \\ & & -1 & \\ & & \dots & \\ (-1)^{n-1} & & & \end{pmatrix} \tag{0.0.0.5}$$

The theorem is clear for unramified generic representations. For general π, one has to prove the equality of traces: $tr(\hat{\pi}) = tr(\tilde{\pi})$ which follows from the fact that a distribution on G, invariant under inner automorphisms, is also invariant under transposition (see B-Z, §7).

Assuming this theorem, it is immediate to prove the third point of 0.0.0.3. In fact, the operation $g \mapsto w^T g^{-1} w^{-1}$ preserves both U and ψ, as can be easily verified.

Following Gelfand, Kazhdan, we look at the restriction of π to the subgroups:

$$P_n = \{\begin{pmatrix} * & \dots & * & * \\ * & \dots & * & * \\ 0 & \dots & 0 & 1 \end{pmatrix}\} \tag{0.0.0.6}$$

$$M_n = \{\begin{pmatrix} 1_{n-1} & * \\ & 1 \end{pmatrix}\} \tag{0.0.0.7}$$

Note that $P_n \simeq M_n \times GL_{n-1}$.

Define: $V_{M,\psi} = V/ < \pi(m)v - \psi(m)v; m \in M_n, v \in V >$ (for $\psi = 1$ and $n = 2$ this is the Jacquet module). This is a representation of P_{n-1}, which we denote by $\Phi^-(\pi)$. Then $(\Phi^-)^{n-1}(\pi)$ is a representation of $P_1 = 1$ on $V_{U,\psi}$ and π is non-generic if and only if $(\Phi^-)^{n-1}(\pi) = 0$.

Proposition 0.0.0.4. $\Phi^-(\pi) = 0$ *if and only if* $\pi|_{M_n} = 1$.

From this, it is easy to proof the first claim of 0.0.0.3. Remark that the functor Φ^- depends on ψ, but the previous proposition is true for any $\psi \neq 1$.

The proof of point 2 can be found in §7 of B-Z, loc.cit.

Point 4 is proven in Zelevinsky, Ann.E.N.S, 13.

0.0.1 Sheaves and representations

In this subsection, we introduce a geometric tool in the study of smooth representations and we prove point 5 of theorem 0.0.0.3.

Let $\pi : G \to GL(V)$ be smooth and irreducible. We look at the restriction of π to $M_n \simeq F^{n-1}$. We have an action of the algebra $C_c^\infty(F^{n-1})$ of complex, locally constant, compactly supported functions on F^{n-1}, on V. It is convenient to take the following modified action:

$$\phi v = \int_{M_n} \hat{\phi}(m)\pi(m)vdm \tag{0.0.1.1}$$

(with some abuse of notation, we identify M_n with F^{n-1}), where

$$\hat{\phi}(y) = \int_{M_n} \phi(z)\psi(y^t z)dz$$

with the self-dual measure dz on $M_n \simeq F^{n-1}$ Then, since $\widehat{\phi_1 * \phi_2} = \hat{\phi}_1\hat{\phi}_2$, we can take on $C_c^\infty(F^{n-1})$ the product of functions, instead of convolution.

We want to prove:

Lemma 0.0.1.1. *For any $v \in V$ there is a open compact subset $U \subset F^{n-1}$ such that $1_U v = v$.*

Proof. Let $\mathfrak{p}^m$ be the conductor of ψ and choose k such that $\pi(\begin{matrix} 1_{n-1} & x \\ & 1 \end{matrix})v = v$ for any $x \in (\mathfrak{p}^k O_F)^{n-1}$. Then:

$$1_{\mathfrak{p}^{-k}} v = \int_{F^{n-1}} \hat{1}_{\mathfrak{p}^{-k}}(x)\pi(\begin{matrix} 1_{n-1} & x \\ & 1 \end{matrix})vdx =$$
$$vol(\mathfrak{p}^{-k}) \int_{F^{n-1}} 1_{\mathfrak{p}^{m+k}}(x)\pi(\begin{matrix} 1_{n-1} & x \\ & 1 \end{matrix})vdx = vol(\mathfrak{p}^{-k})vol(\mathfrak{p}^{m+k})v = v$$

where the last equality holds because of the self-duality of the measure. □

This lemma allows us to attach to π a sheaf on F^{n-1} in the following way. We first define a pre-sheaf by $\mathcal{F}_\pi(U) = 1_U V$, for any open compact subset $U \subset F^{n-1}$. It is not hard to check the sheaf axiom. Moreover, note that:

$$\mathcal{F}_{\pi,x} \simeq V/ < v; 1_U v = 0 \text{ for some U containing x} > \tag{0.0.1.2}$$

Lemma 0.0.1.2. *We have: $\mathcal{F}_{\pi,x} \simeq V_{M_n,\psi}$ for any character ψ. Moreover, $\mathcal{F}_{\pi,x} \simeq \mathcal{F}_{\pi,\mu}$ for two non-trivial characters ψ, μ.*

Proof. We take $n = 2$ for simplicity. Choose $U = x + \mathfrak{p}^k$, k big enough. Then

$$\hat{1}_U(y) = \int_{\mathfrak{p}^k} \psi(y(x+z))dz = \psi(xy)\int_{\mathfrak{p}^k} \psi(yz)dz = \psi(xy)vol(\mathfrak{p}^k)1_{\mathfrak{p}^{m-k}}(y)$$

Therefore $1_U v = 0$ if and only if

$$\int_{\mathfrak{p}^{m-k}} \pi(\begin{matrix} 1 & x \\ & 1 \end{matrix})\psi(xt)dt = 0$$

and the proposition follows from the known characterization of Jacquet modules (indeed, the proof of that characterization applies also when ψ is not trivial).

The last statement follows from the F^*-equivariance of the sheaf $\mathcal{F}_\pi$: the matrix $\begin{pmatrix} a & \\ & 1_{n-1} \end{pmatrix}$ induces an isomorphism $\mathcal{F}_{\pi,\psi} \simeq \mathcal{F}_{\pi,\psi_a}$ by translation, where $\psi_a(x) = \psi(ax)$.

□

As an application, we can prove the following:

Theorem 0.0.1.3. *If $n = 2$, every irreducible, infinite-dimensional representation is generic.*

Proof. If π is not generic, then $V_{M_2,\psi} = 0$ for every $\psi \neq 1$. This means that the only stalk which can be nonzero, is $V_{M_2,1}$. By construction V is the space of compactly supported sections of $\mathcal{F}_\pi$, so that $V \simeq V_{M_2}$ and this means that the unipotent radical $U = M_2$ acts trivially on V. Then the theorem follows from the next proposition. □

Proposition 0.0.1.4. *If $n = 2$ and π is infinite-dimensional, then $V^U = 0$.*

Proof. Let $v \in V^U$. Use the following identity:

$$\begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1/c \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -1/c \\ c & 0 \end{pmatrix} \begin{pmatrix} 1 & 1/c \\ 0 & 1 \end{pmatrix} \tag{0.0.1.3}$$

which implies that $\begin{pmatrix} 0 & -1/c \\ c & 0 \end{pmatrix}$ fixes v if c is small enough. In particular:

$$\begin{pmatrix} a & \\ & 1/a \end{pmatrix} v = \begin{pmatrix} 0 & -a/c \\ c/a & 0 \end{pmatrix} v \tag{0.0.1.4}$$

if c is small enough. Therefore $\begin{pmatrix} a & \\ & 1/a \end{pmatrix} v = v$ for any $a \in F^*$. In turn, this implies:

$$\begin{pmatrix} & 1 \\ -1 & \end{pmatrix} v = \begin{pmatrix} & 1/a \\ -a & \end{pmatrix} v \tag{0.0.1.5}$$

for any $a \neq 0$ and we can choose it small enough. Therefore $SL(2, F)$ fixes v, because of the Bruhat decomposition for $SL(2, F)$:

$$SL(2, F) = B \bigsqcup B \begin{pmatrix} & 1 \\ -1 & \end{pmatrix} B \tag{0.0.1.6}$$

where B is the Borel parabolic. By Schur's lemma, the center of G acts by scalars. But then Gv spans a finite-dimensional space and this contradicts our assumptions.

□

Another application:

Theorem 0.0.1.5. *Take $n = 2$. Let Λ be a nonzero Whittaker functional. Then for any $v \neq 0$ there is a $g \in F^*$ such that $\Lambda(\pi\begin{pmatrix} g & \\ & 1 \end{pmatrix}v) \neq 0$.*

Proof. If $v \mapsto \Lambda(\pi\begin{pmatrix} g & \\ & 1 \end{pmatrix}v) = 0$ for any g, then v goes to zero in $V_{M,\psi}$ for any $\psi \neq 1$. Then $v - \pi(u)v$ has zero image in every $\mathcal{F}_{\pi,x}$, for any $u \in M_2$. This implies $v = 0$, if we use again proposition 0.0.1.4.

□

This theorem implies that V is isomorphic to the space $\mathcal{K}$ of functions on F^* of the form $x \mapsto W_v(\begin{matrix} x & \\ & 1 \end{matrix}), v \in V$. We let G act on $\mathcal{K}$ by right translation. The isomorphism is given by $v \mapsto f_v$, $f_v(g) = W_v(\begin{matrix} x & \\ & 1 \end{matrix})$. The space $\mathcal{K}$ is called the Kirillov model.

For general n, consider the map which associates to $v \in V$ the restriction of W_v to P_n. One can prove that this map is injective. The image is the Kirillov model $\mathcal{K}$. It contains with finite codimension the space $C_c^\infty(P_n, \psi)$ of locally constant, compactly supported functions on P_n with M_n acting as ψ; moreover, $\mathcal{K} = C_c^\infty(P_n, \psi)$ if and only if π is supercuspidal.