L-and M-weakly compact operators

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Introduction

A non-empty bounded subset A of a Banach lattice E is called L-weakly compact if for every disjoint sequence (x_n) in the solid hull of A we have $||x_n|| \to 0$. Every L-weakly compact subset of E is contained in E^a, the largest order ideal in E on which the norm is order continuous. Every relatively compact subset of E^a is L-weakly compact.

Introduction

An operator $T : E \to X$ is called M-weakly compact if $lim_n ||Tx_n|| = 0$ for every norm bounded disjoint sequence in E.
An operator $T : X \to E$ is called L-weakly compact if $T(B_X)$ is an L-weakly compact subset of E.

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Example $T : l^{\infty} \to c_0$ is weakly compact iff it is M-weakly compact iff it is a Dunford- Pettis operator.
If E has order continuous norm or E' is discrete then T is M-weakly compact iff it is weakly compact. By duality, $T : X \to E$ from a Banach space X into an AL-space E is weakly compact iff it is L-weakly compact.

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$T : X \to Y$ is called Dunford-Pettis operator if maps weakly convergent sequences onto norm convergent sequences.
$T : E \to Y$ is called almost Dunford-Pettis if $||Tx_n|| \to 0$ for each weakly null sequence (x_n) of disjoint elements in E_+.
$T : X \to Y$ is called a weak Dunford-Pettis operator whenever $x_n \xrightarrow{w} 0$ in X and $y'_n \xrightarrow{w} 0$ in Y', we have $lim_n(Tx_n, y'_n) = 0$.

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$T : X \to E$ is called semicompact if for each $\epsilon > 0$, there exists some $u \in E_+$ such that $T(B_X) \subseteq [-u, u] + \epsilon B_E$.

Introduction

Plenty of L, M-weakly compact operators. If $1 \leq q \leq p \leq \infty$, then all regular operators from L^p into L^q are L and M-weakly compact. If X does not contain a subspace isomorphic to c_0, then every operator $T : E \to X$ is M-weakly compact. If F is an infinite dimensional L-space, then every regular $T : E \to F$ is L-weakly compact iff E' has order continuous norm[18].

L and M-weakly compact operators

Let $T : E \to X$. Let q_T be the lattice seminorm defined on E as follows:

$$q_T(x) = sup\{||T(y)|| : |y| \leq |x|\}$$

for $x \in E$.

Prop. 1 $T : E \to X$ is M-weakly compact iff $q_T(x_n) \to 0$ for each norm bounded disjoint sequence in E.

L and M-weakly compact operators

L and positive M-weakly compact operators are semicompact.Converse has been studied by Chen et al. and Aqzzouz et al.[3],[11],[12].As the identity on l^∞ shows a semicompact operator need not be L or M-weakly compact.

L and M-weakly compact operators

Prop. 2 a) Let E be a Banach lattice with order continuous norm. If each positive $T : E \to c_0$ is M-weakly compact, then E is a KB-space.
b) If each positive operator $T : E \to l^\infty$ is M-weakly compact, then E is a KB-space.
c) If each positive operator $T : E \to l^\infty$ is M-weakly compact, then E has the $+$ Schur property. Order continuity of E is essential in (a).If $E = l^\infty, F = c_0$, then each $T : E \to F$ is weakly compact and therefore is M-weakly compact but l^∞ is not a KB-space.

L and M-weakly compact operators

CorollaryLet F be σ-Dedekind complete. If each positive weak Dunford-Pettis operator $T : E \to F$ is M-weakly compact then at least one of the following holds.

a) E has positive Schur property.

b) F has order continuous norm.

L and M-weakly compact operators

Dedekind completeness is essential here. If $E = l^{\infty}$ and $F = c$ then each positive weak Dunford-Pettis operator between them is M-weakly compact but E has no Schur, and F has no order continuous norm.

L and M-weakly compact operators

Prop. 3 a) Let $T : E \to X$ be an L-weakly compact operator. Then for each sequence $(g_n) \subset E'$ satisfying $(g_n) \to 0$ for the topology $|\sigma|(E', E)$, we have $||T'(g_n)|| \to o$.
b) If $T' : X' \to E'$ is a M-weakly compact then $||T(x_n)|| \to 0$ for every sequence $(x_n) \subset E$ satisfying $(x_n) \to 0$ for the topology $|\sigma|(E, E')$.

L and M-weakly compact operators

In general L-weakly, M-weakly compact and Dunford-Pettis operators are distict classes of operators. For example, The identity, $I : l^1 \rightarrow l^1$ is a Dunford-Pettis operator but not an L or M-weakly compact. On the other hand, the inclusion map $i : L^2[0,1] \rightarrow L^1[0,1]$ is M-weakly compact but not a Dunford-Pettis operator.

L and M-weakly compact operators

Corollary a) Let $T : E \to F$ be a L-weakly compact operator. If E' has weak* sequentially continuous lattice operations, then T' is Dunford-Pettis,
b) Let $T : E \to F$ be M-weakly compact operator. If E has weakly sequentially continuous lattice operations, then T is Dunford-Pettis.

L and M-weakly compact operators

Each positive Dunford-Pettis operator $T : E \to F$ is L-weakly compact if F is finite dimensional or when $(E, F) \in K$. L and M-weak compactness of positive Dunford-Pettis operators were studied in [5].On the other hand if F has order continuous norm, then each regular L- or M-weakly compact operator is a almost Dunford-Pettis operator.[8]

L and M-weakly compact operators

Prop. 4 a) Let $T : E \to F$ be an L-weakly compact operator, then T' is almost Dunford-Pettis .
b) Let $T : E \to F$ be a positive M-weakly compact, then T' is an almost Dunford-Pettis .
There are almost Dunford-Pettis operators that are not L-weakly or M-weakly compact. The identity $L^1[0,1] \to L^1[0,1]$ is almost Dunford-Pettis but neither M-weakly or L-weakly compact.

L and M-weakly compact operators

Let E be with E' having order continuous norm. Let (x_n) be a norm bounded disjoint sequence in E_+. Then $x_n \stackrel{w}{\to} 0$ in E. As T is a Dunford-Pettis operator, $||Tx_n|| \to 0$ and T is M-weakly compact.Weak Dunford-Pettis operators need not be M-weakly compact.

L and M-weakly compact operators

Example Identity on c_0 is a weak Dunford-Pettis operator as c_0 has the weak Dunford-Pettis property, but it is not M-weakly compact. On the other hand each operator T from l^∞ to c is M-weakly compact as l^∞ has order continuous dual and has the Dunford-Pettis property.
The following result discusses M-weak compactness of weak Dunford-Pettis operators.

L and M-weakly compact operators

Prop.5 Suppose E' is a KB-space. Let $T : E \to F$ be a positive weak Dunford-Pettis operator. Each of the following imply that T is M-weakly compact.

1) F is a dual KB-space.
2) F is a discrete KB-space.
3) F'' has order continuous norm.
4) E has the Schur property.

L and M-weakly compact operators

None of the preceding is sufficient. For example, $E = l^\infty$ and $F = c_0$. Then since E' has order continuous norm each operator $T : E \to F$ is Dunford-Pettis and therefore M-weakly compact but c_0 is not a KB-space, l^∞ does not have the Schur property, and $(c_0)'' = l^\infty$ has no order continuous norm. In particular if E' is a KB-space, then each of the above yields aDPO = wDPO.[15]

L and M-weakly compact operators

Prop. 6 Suppose E has weakly sequentially continuous lattice operations. Then each regular L-weakly compact operator $T : E \to F$ is a Dunford-Pettis operator for each Banach lattice F. Theorem 2.26 in [5] shows that a positive Dunford-Pettis operator between Banach lattices E and F is weakly compact if and only if one of a) E' has order continuous norm or b) F is reflexive.

L and M-weakly compact operators

We generalize this and show that under the same conditions each Dunford-Pettis operator is L or M-weakly compact.

Prop.7 The following are equivalent:

1) Each positive Dunford-Pettis operator is either L-weakly or M-weakly compact.

2) One of the following holds.

a) E' has order continuous norm

b) F is reflexive.

L and M-weakly compact operators

Prop. 8 Let E' be a KB-space and F be an AL-space.Let $T : E \to F$ be a regular operator such that $T[0, x]$ is $|\sigma|(F, F')$-totally bounded for each $x \in E_+$. Then T is M-weakly compact.

Corollary Let E, F be as above and $T : E \to F$ be a regular AM-compact operator. Then T is M-weakly compact.

L and M-weakly compact operators

There are M-weakly compact operators that are not AM-compact. Consider the natural embedding $J : L^{\infty}[0,1] \to L^{p}[0,1]$ for $p, 1 \leq p \leq \infty$. J is M-weakly compact but not AM-compact.

If E is a Banach lattice with the Dunford-Pettis property then L- and M-weakly compact operators from E to an arbitrary X is a Dunford-Pettis operator. The identity on $L^{1}[0,1]$ is a weak Dunford-Pettis operator which is not L or M-weakly compact.

L and M-weakly compact operators

Proposition 9 Suppose each positive AM-compact operator $T : E \to F$ is L-weakly compact, then one of E' or F' has order continuous norm.

L and M-weakly compact operators

Proposition 10 Let E be a Banach lattice with order continuous norm. Suppose each positive AM-compact operator $T : E \to F$ is M-weakly compact. Then one of E or F is a KB-space.

Finite ranks determine order

Rank-one operators may and do determine quite a lot of structure of the underlying space. Suppose a Banach space X satisfies the Daugavet property for rank-one operators $||I + T|| = 1 + ||T||$ for each rank-one T, then X contains l^1.If a Banach space X satisfies the Daugavet property for rank-one operators, then X can not have an unconditional basis.

Finite ranks determine order

Suppose for any two rank-one operators T_1, T_2 from E to F, we have $||T_1 \vee T_2|| = ||T_1|| \vee ||T_2||$, then E is an AL-space, F is an AM-space.

Finite ranks determine order

A finite rank operator need not be an M-weakly or L-weakly compact.

Example Consider $T : l^1 \to l^\infty$, defined by $T(\alpha_n) = (\sum_n \alpha_n)e$ for each $(\alpha_n) \in l^1$ where e is the constantly one sequence in l^∞.

Finite ranks determine order

Example When E is finite dimensional, then all operators on E are M-weakly compact.

Prop. 1 Let E, F be non-zero Banach lattices. Each positive rank-one operator $T : E \to F$ is L-weakly compact if and only if F has order continuous norm.

Finite ranks determine order

Corollary The following are equivalent:
1) Each positive rank-one operator $T : E \to F$ is L-weakly compact.
2) Each positive semicompact operator $T : E \to F$ is L-weakly compact.
3) Each positive compact operator $T : E \to F$ is L-weakly compact.

Finite ranks determine order

Corollary A non-zero Banach lattice E has order continuous norm iff every approximable operator on E is L-weakly compact.

Corollary If F is an non-atomic Banach lattice with order continuous norm then every $+$ finite rank operator is disjoint to the center $Z(E)$ of E.

Finite ranks determine order Finite ranks determine order

Corollary Let F be a Banach lattice with order continuous norm, then each discrete operator in $L_b(E, F)$ is L-weakly compact.
Corollary Let E, F be Banach lattices. The dual E' of E has order continuous norm iff every $+$ rank-one operator $T : E \to F$ is M-weakly compact.

Finite ranks determine order

Corollary Let E, F be non-zero Banach lattices. TFAE.

1) Each positive rank-one operator $T : E \to F$ is M-weakly compact.

2) Each finite rank operator $T : E \to F$ is M-weakly compact.

3) E' has order continuous norm.

Corollary Let E' have order continuous norm. Then each discrete operator in $L_b(E, F)$ is M-weakly compact.

Finite ranks determine order

Corollary Let E, F be non-zero Banach lattices. Each $+$ finite rank operator is L-weakly and M-weakly compact iff $(E, F) \in K$

Let E, F be Banach lattices where F is Dedekind complete. An order bounded operator $T : E \to F$ has order continuous norm if for each sequence (T_n) of $+$ operators with $|T| \geq T_n \downarrow 0$ in $L_b(E, F)$, we have $||T_n|| \to 0$.$+$ operators that have order continuous norms are precisely the $+$ operators which are simultaneously L-weakly and M-weakly compact operators.

Finite ranks determine order

Equivalence of (1),(2), (4) in the following result was proved in [5].

Corollary Let E, F be non-zero Banach lattices.TFAE: 1) Each positive Dunford-Pettis $T : E \to F$ is M-weakly compact.

2) Each positive compact $T : E \to F$ is M-weakly compact.

3) Each rank-one operator $T : E \to F$ is M-weakly compact.

4) E' has order continuous norm.

Finite ranks determine order

Corollary TFAE:

1) Each + rank-one $T : E \to F$ has order continuous norm.

2) Each + Dunford-Pettis $T : E \to F$ has order continuous norm.

3) Each + compact $T : E \to F$ has order continuous norm.

4) $(E, F) \in K$.

Finite ranks determine order

If $T : E \to F$ is an order bounded operator. The order ideal generated by T in $L_b(E, F)$ is denoted by A_T.

Corollary Let E be a Banach lattice which is either σ-Dedekind complete or has a quasi-interior point and F be Dedekind complete. If $(E, F) \in K$, then for each finite rank T, $A_T \subseteq ring(T)$.

Finite ranks determine order

Prop.2 For a Banach lattice E TFAE:

1) Each + finite rank T and operator S with $0 \leq S \leq T$, S is M- weakly compact.
2) Each positive finite rank operator T on E is M-weakly compact.
3) For each positive finite rank operator T on E, T^2 is M-weakly compact.
4) E' has order continuous norm.

Finite ranks determine order

The following is dual to the preceding result.

Prop.3. For a Banach lattice E TFAE:

1) For each positive finite rank operator T and operator S with $0 \leq S \leq T$, S is L- weakly compact.

2) Each positive finite rank operator T on E is L-weakly compact.

3) For each positive finite rank operator T on E, T^2 is L-weakly compact.

4) E has order continuous norm.

Finite ranks determine order

Prop. 4 For a pair of Banach spaces E, F TFAE:
1) At least one of the following holds:
a) F has order continuous norm.
b) $(E')^a = 0$.
2) Each regular M-weakly compact operator $T : E \to F$ is L-weakly compact.
3) Every rank-one M-weakly compact operator $T : E \to F$ is L-weakly compact.

Finite ranks determine order

Prop. 5TFAE for E and F:
1) a) E' has order continuous norm.
b) $F^a = 0$.
2) Each regular L-weakly compact operator $T : E \to F$ is M-weakly compact.
3) Every L-weakly compact rank-one operator $T : E \to F$ is M-weakly compact.

Finite ranks determine order

If E' has the $+$ Schur property, then E' has order continuous norm and thus the class of operators of strong type B, $W_{st}(E,X)$ coincides with weakly compact operators $W(E,X)$. In this case not only every weakly compact operator is M-weakly compact but the larger class of operators of strong type B, are M-weakly compact. Thus theorem 3.3. in [11] takes the following form.

Finite ranks determine order

Prop.6 The following are equivalent for a Banach lattice E.

1) E' has the positive Schur property.

2) For each Banach space X, every operator of strong type B, $T : E \to X$ is M-weakly compact.

3) Every positive weakly compact operator $T : E \to c_0$ is M-weakly compact.

Finite ranks determine order

Prop.7 Let F be a Banach lattice with order continuous norm. If $T : E \to F$ is an disjointness preserving operator that is dominated by a compact operator S.i.e., $|T(x)| \leq |S|(|x|)$ for each $x \in E$, then T is M-weakly compact.

Finite ranks determine order

An operator is called bounded from below if there exists some δ such that $\delta||x|| \leq ||T(x)||$ for all x. Strictly singular and M-weakly compact operators lie outside the open subset of operators that are bounded below. The next example shows that strictly singular and M-weakly compact operators may be distinct. **Example** Let $1 \leq p < r < \infty$. Then the canonical embedding $J : l^p \to l^r$ is strictly singular but not M-weakly or L-weakly compact operators as $||e_n - e_m|| = 2^{1/r}$.

b-weakly compact operators

Prop.1 Each + operator $T : E \to l^\infty$ is b-weakly compact if and only if E is a KB-space.
Let $T : E \to X$ be a Dunford-Pettis operator and let (x_n) be a positive b-bounded disjoint sequence in E. Then $x_n \xrightarrow{w} 0$, hence $||Tx_n|| \to 0$ and T is a b-weakly compact operator. The next result gives a sufficient condition for the larger class of weak Dunford-Pettis operators to be contained in $W_b(E, F)$.

b-weakly compact operators

Prop.2 Let F be a dual Banach lattice with order continuous norm. Then each positive weak Dunford-Pettis operator $T : E \to F$ is b-weakly compact.

b-weakly compact operators

Corollary If the range F is a discrete KB-space then each $+$ weak Dunford-Pettis operator is b-weakly compact.

The following is a converse to the previous result.

Prop.3 Let F be Dedekind σ-complete. If each $+$ weak Dunford-Pettis operator $T : E \to F$ is b-weakly compact. Then at least one of the following holds.

a) F has order continuous norm.

b) E is a KB-space.

b-weakly compact operators

Corollary Let F be a Dedekind σ- complete Banach lattice whose norm is not order continuous. TFAE:

a) Each positive operator $E \to F$ is b-weakly compact.

b) E is a KB-space.

b-weakly compact operators

In Prop. 3 the necessary condition that E has order continuous norm is not sufficient. The identity on c_0 is a weak Dunford-Pettis operator as c_0 has the Dunford-Pettis property but it is not b-weakly compact. The assumption that F is Dedekind σ-complete is essential above. For if $E = l^\infty$ and $F = c$, then c is not Dedekind σ-complete. However, each operator from E into F is Dunford-Pettis and therefore b-weakly compact. But E is not a KB-space.

Phillips property

Given a Banach space Y and an operator $T : X'' \to Y$, let $\hat{T} : X \to Y$ denote the restriction of T to X.

We identify a Banach space X with its image under the canonical embedding of X in X''. Then there exists canonical projection $p : X''' \to X'$ corresponding to the direct sum decomposition $X''' = X' \oplus X^{\perp}$.

Definition X is said to have the Phillips property if $p : X''' \to X'$ is sequentially weak* to norm continuous

Phillips property

Prop.1 A Banach lattice E has the Phillips property if and only if for every operator $T : E'' \to c_0$, the operator $\hat{T} : E \to c_0$ is L-weakly compact.

Phillips property

Corollary 2.8 in [11] shows us that every L-weakly compact operator from a Banach space into an AM-space is compact. Utilizing this we obtain the following result of Ulger and Freedman.

Corollary A Banach lattice E has the Phillips property if and only if for every operator $T : E'' \to c_0$, the operator $\hat{T} : E \to c_0$ is compact.

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