On positive linear operators preserving polynomials

Francesco Altomare, University of Bari, Italy

Positivity VII

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The talk reports the main results of the joint paper

- **F. Altomare, M. Cappelletti Montano, V. Leonessa and I. Raşa,**
  On Markov operators preserving polynomials,
  preprint, **2013**.

The title refers to a special class of

positive linear operators

acting on the space $C(K)$ of all continuous functions defined on a

convex compact subset K of $\mathbf{R}^d$, $d \geq 1$,

having non-empty interior.

More precisely, denote by $\mathbf{1}$ the constant function 1 on K and, for every $i \in \{1, \ldots, d\}$, by pr_i the i-th **coordinate function** on K, i.e.

$$pr_i(x) = x_i \text{ for every } x = (x_1, \ldots, x_d) \in K.$$

For every $m \geq 1$, we denote by $P_m(K)$ the linear subspace of the

(restriction to K of the) **polynomials of degree no greater than** m.

We are interested in those **Markov linear operator**

$$T : C(K) \to C(K),$$

i.e., T is positive and $T(1) = 1$, satisfying

$$T(h) = h \qquad \text{for every } h \in \{\mathbf{1}, pr_1, \ldots, pr_d\}, \tag{1.1}$$

i.e., T **leaves invariant polynomials of degree at most 1**

and

$$T(P_m(K)) \subset P_m(K) \qquad \text{for every } m \geq 2. \tag{1.2}$$

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Within this class, a special role is played by those Markov operators T which in addition are **positive projections**, i.e.,

$$T^2 := T \circ T = T$$

and such that their range

$$H := T(C(K)) = \{f \in C(K) \mid T(f) = f\}$$

are **invariant under affine transformations**, i.e.,

$$h \circ \sigma_{z,\alpha} \in H \qquad \text{for every } h \in H, z \in K \text{ and } \alpha \in [0,1]$$

where

$$\sigma_{z,\alpha}(x) = \alpha x + (1-\alpha)z \text{ for every } x \in K.$$

Such positive projections will be referred to as **A-projections**.

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The interest for such operators comes from the study of a special differential operator $(W_T, C^2(K))$ which can be associated with a Markov operator T and which is defined as

$$W_T(u) := \frac{1}{2} \sum_{i,j=1}^{d} \alpha_{ij} \frac{\partial^2 u}{\partial x_i \partial x_j}$$

$(u \in C^2(K))$, where

$$\alpha_{ij} := T(pr_i pr_j) - (pr_i pr_j) \ (i, j = 1, \ldots, d).$$

The differential operator W_T has been carefully investigated in

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The differential operator W_T is elliptic and it degenerates on a subset of K which contains the set of the extreme points $\partial_e K$ of K.

In the above mentioned paper we showed that, if T maps polynomials into polynomials of the same degree, then $(W_T, C^2(K))$ **is closable in** $C(K)$ **and its closure generates a Markov semigroup on** $C(K)$ which can be represented as a **limit of suitable iterates of particular positive linear operators associated with** T**, namely the Bernstein-Schnabl operators associated with** T. Next we proceed to discuss such a generation result in more details.

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2. Some preliminaries on Markov operators

A useful tool we shall use in the sequel is the notion of

Choquet boundary.

Given a linear subspace H of $C(K)$, the *Choquet boundary* of H is the subset of all points $x \in K$ such that,

$$\text{if } \tilde{\mu} \in M^+(K) \text{ and if } \int h\, d\tilde{\mu} = h(x) \text{ for every } h \in H,$$

then

$$\int f\, d\tilde{\mu} = f(x) \text{ for every } f \in C(K).$$

It will be denoted by

$$\partial_H K.$$

If H contains the constants and separates the points of K, then $\partial_H K$ is non-empty and every $h \in H$ attains its minimum and maximum on $\partial_H K$. Therefore,

$$\text{if } f, g \in H \text{ and if } f = g \text{ on } \partial_H K, \text{ then } f = g \text{ on } K.$$

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An important example of Choquet boundary is the set

$$\partial_e K$$

of the **extreme points** of K.

They are defined as those points $x_0 \in K$ such that $K \setminus \{x_0\}$ is convex.

Indeed, denote by $P_1(K)$ the space of (the restriction to K of) all polynomials of degree at most 1. Clearly, $P_1(K)$ contains the constants and separates the points of K.

As a matter of fact, it turns out that

$$\partial_{P_1(K)} K = \partial_e K. \tag{1.3}$$

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Now let us consider a Markov operator $T : C(K) \to C(K)$ and set

$$M := \{h \in C(K) \mid T(h) = h\}. \tag{1.4}$$

Clearly, M is contained in the range of T which will be denoted by

$$H := T(C(K)) = \{T(f) \mid f \in C(K)\}. \tag{1.5}$$

The subspace M contains the constants and hence, if it separates the points of K, then its Choquet boundary $\partial_M K$ is non-empty. In the sequel, the following subset

$$\partial_T K := \{x \in K \mid T(f)(x) = f(x) \text{ for every } f \in C(K)\} \tag{1.6}$$

will play an important role. Its elements are also called the

interpolation points of the operator T.

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Theorem 2.1

Consider a Markov operator $T : C(K) \to C(K)$ such that the subspace M separates the points of K. Then

$$\emptyset \neq \partial_M K \subset \partial_T K \subset \partial_H K. \tag{1.7}$$

Moreover, if V is an arbitrary subset of M separating the points of K,

$$\partial_T K = \{x \in K \mid T(h^2)(x) = h^2(x) \text{ for every } h \in V\}. \tag{1.8}$$

Finally, if $pr_i \in M$, *i.e.*, $T(pr_i) = pr_i$ for every $i = 1, ..., d$, then

$$\Phi \leq T(\Phi),$$

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Proposition 2.2

The following statements are equivalent:

(a) T is a projection, i.e., $T^2(f) = T(f)$ for every $f \in C(K)$.

(b) There exists a subset V of M separating the points of K such that $T^2(h^2) = T(h^2)$ for every $h \in V$, i.e., $T(V^2) \subset M$.

Moreover, if $T(pr_i) = pr_i$ for every $i = 1, ..., d$, then statement (a) and (b) are equivalent to

(c) $T^2(\Phi) = T(\Phi)$, where again $\Phi := \sum_{i=1}^{d} pr_i^2 = \| \bullet \|^2$.

Moreover, if (a), (b) or (c) holds true, then $M = H$ and hence

$$\partial_M K = \partial_T K = \partial_H K.$$

If in addition T is an A-projection, then $\partial_T K \subset \partial K$. Finally, for every $f, g \in C(K)$,

$$T(f) = T(g) \quad \textit{provided } f = g \textit{ on } \partial_H K. \tag{1.10}$$

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We recall that a **simplex** of $\mathbf{R}^d$ is the convex hull of some $d+1$ affinely independent points of $\mathbf{R}^d$.

Therefore, the subset

$$K_d := \left\{ (x_1, \ldots, x_d) \in \mathbf{R}^d \;\middle|\; x_i \geq 0 \text{ for every } i = 1, \ldots, d \text{ and } \sum_{i=1}^{d} x_i \leq 1 \right\} \tag{1.11}$$

being the convex hull of $\{v_0, \ldots, v_d\}$, where

$$v_0 := (0, \ldots, 0), v_1 := (1, 0, \ldots, 0), \ldots, v_d := (0, \ldots, 0, 1), \tag{1.12}$$

is a simplex in $\mathbf{R}^d$ and it is called the **canonical simplex** of $\mathbf{R}^d$.

Note that,

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According to the next theorem, when K is a simplex, then on $C(K)$ there exists a (unique) natural positive projection T on $C(K)$ such that $T(C(K)) = P_1(K)$ (**F. A. (1977)**)

Theorem 2.3

Given a convex compact subset K of $\mathbf{R}^d$, $d \geq 1$, the following statements are equivalent:

(a) K is a simplex.

(b) For every $x \in K$ there exists a unique $\tilde{\mu}_x \in M_1^+(K)$ such that $\tilde{\mu}_x(K \setminus \overline{\partial_e K}) = 0$ and

$$\int_K h \, d\tilde{\mu}_x = h(x) \qquad \text{for every } h \in P_1(K).$$

(c) Every continuous function $f : \partial_e K \longrightarrow \mathbf{R}$ can be continuously extended to a (unique) function $\tilde{f} \in P_1(K)$.

(d) There exists a (unique) positive projection $T : C(K) \longrightarrow C(K)$ such that $T(C(K)) = P_1(K)$.

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(d) There exists a (unique) positive projection $T : C(K) \longrightarrow C(K)$ such that $T(C(K)) = P_1(K)$.

Moreover, if one of these statements holds true, then for every $f \in C(K)$ and $x \in K$,

$$T(f)(x) = \int_K f \, d\tilde{\mu}_x = \widetilde{f|_{\partial_e K}}(x). \tag{1.13}$$

Given a simplex K of $\mathbf{R}^d$, the positive projection $T : C(K) \longrightarrow C(K)$ as in condition (d) is referred to as **the canonical positive projection** associated with K.

Thus, for every $f \in C(K)$, $T(f)$ is the unique continuous affine function on K that coincides with f on $\partial_e K$.

In the case $K = K_d$, $d \geq 1$, the canonical projection is given by

$$T_d(f)(x) := \left(1 - \sum_{i=1}^{d} x_i\right) f(v_0) + \sum_{i=1}^{d} x_i f(v_i) \tag{1.14}$$

$(f \in C(K_d), x = (x_1, \ldots, x_d) \in K_d, v_0, \ldots, v_d$ as in (1.12).

In particular, for $d = 1$,

$$T_1(f)(x) := (1 - x)f(0) + xf(1) \tag{1.15}$$

$(f \in C([0, 1]), 0 \leq x \leq 1)$.

Moreover, if one of these statements holds true, then for every $f \in C(K)$ and $x \in K$,

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Given a simplex K of $\mathbf{R}^d$, the positive projection $T : C(K) \longrightarrow C(K)$ as in condition (d) is referred to as **the canonical positive projection** associated with K.

Thus, for every $f \in C(K)$, $T(f)$ is the unique continuous affine function on K that coincides with f on $\partial_e K$.

In the case $K = K_d$, $d \geq 1$, the canonical projection is given by

$$T_d(f)(x) := \left(1 - \sum_{i=1}^{d} x_i\right) f(v_0) + \sum_{i=1}^{d} x_i f(v_i) \tag{1.14}$$

$(f \in C(K_d), x = (x_1, \ldots, x_d) \in K_d, v_0, \ldots, v_d$ as in (1.12).

In particular, for $d = 1$,

$$T_1(f)(x) := (1 - x)f(0) + xf(1) \tag{1.15}$$

$(f \in C([0,1]), 0 \leq x \leq 1)$.

Moreover, if one of these statements holds true, then for every $f \in C(K)$ and $x \in K$,

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3. An approximation process

Given a Markov operator $T : C(K) \to C(K)$, by the Riesz representation theorem there exists a unique family $(\tilde{\mu}_x^T)_{x \in K}$ in $M_1^+(K)$ such that

$$T(f)(x) = \int_K f \, d\tilde{\mu}_x^T \qquad (f \in C(K), x \in K). \tag{1.16}$$

Such a family is said to be the **continuous selection of probability Borel measures on K associated with T**.

By means of $(\tilde{\mu}_x^T)_{x \in K}$ we can construct the so-called **Bernstein-Schnabl operators associated with T** which are defined by setting, for every $n \geq 1$, $x \in K$ and $f \in C(K)$,

$$B_n(f)(x) = \int_K \cdots \int_K f\Big(\frac{x_1 + \ldots + x_n}{n}\Big) d\tilde{\mu}_x^T(x_1) \cdots d\tilde{\mu}_x^T(x_n). \tag{1.17}$$

Note that by the continuity property of the product measure it follows that $B_n(f) \in C(K)$. Moreover, $B_1 = T$.

Given a Markov operator $T : C(K) \to C(K)$, by the Riesz representation theorem there exists a unique family $(\tilde{\mu}_x^T)_{x\in K}$ in $M_1^+(K)$ such that

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Note that by the continuity property of the product measure it follows that $B_n(f) \in C(K)$. Moreover, $B_1 = T$.

For $K = K_d$ and $T = T_d$, then the B_n's become the classical Bernstein operators on $C(K_d)$:

$$B_n(f)(x) := \sum_{\substack{h_1,\dots,h_p=0,\dots,n \\ h_1+\dots+h_p \le n}} f\left(\frac{h_1}{n},\dots,\frac{h_p}{n}\right) \frac{n!}{h_1!\dots h_p!(n-h_1-\dots-h_p)!}$$

$$\times\ x_1^{h_1}\dots x_p^{h_p}\left(1-\sum_{i=1}^{p} x_i\right)^{n-\sum\limits_{i=1}^{p} h_i}.$$

For $d = 1$, they turn into

$$B_n(f)(x) := \sum_{k=0}^{n} \binom{n}{k} f\left(\frac{k}{n}\right) x^k(1-x)^{n-k}$$

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For a comprehensive survey on these operators (including noteworthy examples), we refer to

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and to the references contained in the relevant notes. Here we only point out that

$$B_n(f) = f \text{ on } \partial_T K \text{ for every } f \in C(K) \tag{1.18}$$

and, if in addition the Markov operator T satisfies

$$T(h) = h \qquad \text{for every } h \in \{\mathbf{1}, pr_1, \ldots, pr_d\}, \tag{1.19}$$

then the sequence $(B_n)_{n\geq 1}$ is a positive approximation process in $C(K)$, that is

$$\lim_{n\to\infty} B_n(f) = f \quad \text{uniformly on } K \text{ for every } f \in C(K). \tag{1.20}$$

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4.Differential operators associated with Markov operators

From now on fix a Markov operator $T : C(K) \longrightarrow C(K)$ satisfying

$$T(h) = h \qquad \text{for every } h \in \{\mathbf{1}, pr_1, \ldots, pr_d\},$$

K being a convex compact subset $\mathbf{R}^d$, $d \geq 1$, whose interior is assumed to be non-empty.

Consider the differential operator $W_T : C^2(K) \longrightarrow C(K)$ defined by

$$W_T(u) := \frac{1}{2} \sum_{i,j=1}^{d} \alpha_{ij} \frac{\partial^2 u}{\partial x_i \partial x_j} \tag{1.21}$$

$(u \in C^2(K))$, where, for each $i, j = 1, \ldots, d$

$$\alpha_{ij} := T(pr_i pr_j) - (pr_i pr_j)). \tag{1.22}$$

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Accordingly, if $\xi_1, \ldots, \xi_d \in \mathbf{R}$, then

$$\sum_{i,j=1}^{d} \alpha_{ij}(x)\xi_i\xi_j = T\left(\left(\sum_{i=1}^{d} \xi_i(pr_i - x_i)\right)^2\right)(x) \geq 0,$$

which implies that W_T is elliptic.

Moreover, it degenerates on $\partial_T K$ and, in particular, on $\partial_e K$ because $\alpha_{ij} = 0$ on $\partial_T K$ for every $i, j = 1, \ldots, d$.

The operator W_T will be referred to as the **elliptic second order differential operator associated with the Markov operator T.** Note also that for each $i, j = 1, \ldots, d$

$$W_T(pr_i pr_j) = \alpha_{ij} = T(pr_i pr_j) - pr_i pr_j$$

and hence, if $P \in P_2(K)$, then $W_T(P) = T(P) - P$.

Therefore, if T is a Markov projection and $T(P_2(K)) \subset P_2(K)$, then

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Differential operators of the form (1.20) are of concern in the study of diffusion problems arising from different areas such as biology, mathematical finance, physics.
In the special case where T is a positive projection, a rather complete overview on them can be found in Chapter 6 of the monograph by F. Altomare - M. Campiti (1994).

It turns out that the differential operator W_T is generated by an asymptotic formula for Bernstein-Schnabl operators.

Theorem 2.4

For every $u \in C^2(K)$,

$$\lim_{n\to\infty} n(B_n(u) - u) = W_T(u) \qquad \text{uniformly on } K. \tag{1.24}$$

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Before stating the next result, we recall that a **core** for a linear operator $A : D(A) \to C(K)$ is a linear subspace D_0 of $D(A)$ which is dense in $D(A)$ with respect to the graph norm

$$\|u\|_A := \|A(u)\|_\infty + \|u\|_\infty (u \in D(A)).$$

Theorem 2.5

Consider a Markov operator T on $C(K)$ which leaves invariant polynomials of degree at most 1 and which maps polynomials into polynomials of the same degree, i.e.,

$$T(P_m(K)) \subset P_m(K) \qquad \text{for every } m \geq 2. \tag{1.25}$$

Then, the differential operator $(W_T, C^2(K))$ is closable and its closure $(A_T, D(A_T))$ generates a Markov semigroup $(T(t))_{t\geq 0}$ on $C(K)$ such that for every $t \geq 0$ and for every sequence $(k(n))_{n\geq 1}$ of positive integers satisfying $\lim_{n\to\infty} k(n)/n = t$, one gets

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Moreover,

$$P_\infty(K) := \bigcup_{m=1}^{\infty} P_m(K) \text{ is a core for } (A_T, D(A_T));$$

if $u, v \in C(K)$ and if $\lim_{n\to\infty} n(B_n(u) - u) = v$ uniformly on K, then

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In particular, if $\lim_{n\to\infty} n(B_n(u) - u) = 0$ uniformly on K, then

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Furthermore,

$$T(t)(f) = f \text{ on } \partial_T K \quad \text{for every } t > 0 \text{ and } f \in C(K). \tag{1.27}$$

and, finally, if T is a projection, then

$$\lim_{t\to+\infty} T(t)f = T(f),$$

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The representation formula (1.26) can be useful to investigate several qualitative and quantitative properties of both the semigroups $(T(t))_{t\geq 0}$ (i.e., of **the solutions to the initial-boundary value problems** associated with the generator A_T) and the **transition functions of the corresponding Markov processes**.

$$\begin{cases} \dfrac{\partial u(x,t)}{\partial t} = A_T(u(\cdot,t))(x), & (x \in K,\ t > 0) \\ \\ u(x,0) = u_0(x), & u_0 \in D(A_T), \end{cases} \tag{1.28}$$

which, as it is well-known, are given by

$$u(x,t) = T(t)(u_0)(x) \qquad (x \in K,\ t > 0). \tag{1.29}$$

Note also that the boundary conditions for problem (1.26) are incorporated in the domain $D(A_T)$. They include the so-called **Wentcel's boundary conditions**

$$A_T u = 0 \quad \text{on } \partial_T K \quad (u \in D(A_T)) \tag{1.30}$$

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Examples

1. Consider a Markov operator T on $C([0,1])$ satisfying (1.1), i.e.,

$$T(e_1) = e_1, \tag{1.31}$$

where $e_1(x) := x$ $(0 \leq x \leq 1)$.
Then, for every $u \in C^2([0,1])$ and $x \in [0,1]$,

$$W_T(u)(x) = \frac{\alpha(x)}{2} u''(x), \tag{1.32}$$

with

$$\alpha(x) := T(e_2)(x) - x^2 \tag{1.33}$$

and $e_2(x) := x^2$ $(0 \leq x \leq 1)$.

Examples of Markov operators on $C([0,1])$ which, in addition, satisfy (1.25) can be easily achieved.
Consider, for instance, for a given $n \geq 1$, the n-th Bernstein operator

$$B_n(f)(x) := \sum_{k=0}^{n} \binom{n}{k} f\left(\frac{k}{n}\right) x^k (1-x)^{n-k}$$

$(f \in C([0,1]), 0 \leq x \leq 1)$.
In this case

$$\alpha(x) = \frac{x(1-x)}{n}$$

$(0 \leq x \leq 1)$.

Examples

2. The differential operator associated with the canonical projection T_d on the d-dimensional simplex K_d is given by

$$\begin{aligned} W_{T_d}(u)(x) &:= \frac{1}{2}\sum_{i,j=1}^{d} x_i(\delta_{ij} - x_j)\frac{\partial^2 u}{\partial x_i \partial x_j}(x) \\ &= \frac{1}{2}\sum_{i=1}^{d} x_i(1 - x_i)\frac{\partial^2 u}{\partial x_i^2}(x) - \sum_{1\le i<j\le d} x_i x_j \frac{\partial^2 u}{\partial x_i \partial x_j}(x) \end{aligned} \tag{1.34}$$

$(u \in C^2(K_d), x = (x_1, \ldots, x_d) \in K_d)$, where δ_{ij} stands for the Kronecker symbol.

The operator (1.34) falls into the class of the so called Fleming-Viot operators.Moreover, the coefficients of W_{T_d} vanish on the vertices of the simplex. In this case

$$T_d(P_m(K_d)) \subset P_1(K_d) \text{ for every } m \geq 2$$

and hence (1.25) holds true.

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$(u \in C^2(K_d), x = (x_1, \ldots, x_d) \in K_d)$, where δ_{ij} stands for the Kronecker symbol.
The operator (1.34) falls into the class of the so called Fleming-Viot operators.Moreover, the coefficients of W_{T_d} vanish on the vertices of the simplex. In this case

$$T_d(P_m(K_d)) \subset P_1(K_d) \text{ for every } m \ge 2$$

and hence (1.25) holds true

Examples

2. The differential operator associated with the canonical projection T_d on the d-dimensional simplex K_d is given by

$$\begin{aligned} W_{T_d}(u)(x) &:= \frac{1}{2}\sum_{i,j=1}^{d} x_i(\delta_{ij} - x_j)\frac{\partial^2 u}{\partial x_i \partial x_j}(x) \\ &= \frac{1}{2}\sum_{i=1}^{d} x_i(1-x_i)\frac{\partial^2 u}{\partial x_i^2}(x) - \sum_{1\le i<j\le d} x_i x_j \frac{\partial^2 u}{\partial x_i \partial x_j}(x) \end{aligned} \tag{1.34}$$

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Consider a symmetric matrix $(a_{ij})_{1 \le i,j \le d}$ of **Hölder continuous functions** on $\mathrm{int}(K)$ with exponent $\beta \in]0, 1[$.
Let L be the differential operator

$$L(u)(x) := \sum_{i,j=1}^{d} a_{ij}(x) \frac{\partial^2 u(x)}{\partial x_i \partial x_j} \tag{1.35}$$

$(u \in C^2(\mathrm{int}(K)), x \in \mathrm{int}(K))$ and assume that it is **strictly elliptic**, i.e., for every $x \in \mathrm{int}(K)$ the matrix $(a_{i,j}(x))_{1 \le i,j \le d}$ is positive-definite and, denoted by $\sigma(x)$ its smallest eigenvalue, we have $\sigma(x) \ge \sigma_0 > 0$, for some $\sigma_0 \in \mathbf{R}$.

Denote by $T_L : C(K) \longrightarrow C(K)$ the **Poisson operator associated with L.**
Thus, for every $f \in C(K)$, $T_L(f)$ denotes the unique solution to the **Dirichlet problem**

$$\begin{cases} Lu = 0 & \text{on } \mathrm{int}(K), \quad u \in C(K) \cap C^2(\mathrm{int}(K)); \\ u = f & \text{on } \partial K. \end{cases} \tag{1.36}$$

T_L is a Markov projection satisfying (1.1) and

$$\partial_T K = \partial K.$$

Consider a convex compact subset K of $\mathbf{R}^d$, $d \geq 2$, such that its boundary ∂K is an ellipsoid, i.e., there exist a real symmetric and positive-definite matrix $R = (r_{ij})_{1 \leq i,j \leq d}$ and $\overline{x} = (\overline{x}_i)_{1 \leq i \leq d} \in \mathbf{R}^d$ such that

$$K = \left\{ x \in \mathbf{R}^d \;\middle|\; Q(x - \overline{x}) := \sum_{i,j=1}^{d} r_{ij}(x_i - \overline{x}_i)(x_j - \overline{x}_j) \leq 1 \right\}. \quad (1.37)$$

Furthermore, consider a strictly elliptic differential operator

$$L(u)(x) := \sum_{i,j=1}^{d} c_{ij} \frac{\partial^2 u}{\partial x_i \partial x_j}(x) \quad (1.38)$$

$(u \in C^2(\mathrm{int}(K)), x \in \mathrm{int}(K))$ associated with a real symmetric and positive-definite matrix $C = (c_{ij})_{1 \leq i,j \leq d}$ and denote by T_L the relevant Poisson operator on $C(K)$.

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Theorem 2.6

Let K and L be as in (1.37) and (1.38). Assume for the sake of simplicity that

$$\sum_{i,j=1}^{d} r_{ij}c_{ij} = 1.$$

Then the differential operator W_L associated with T_L is given by

$$W_L(u)(x) = \begin{cases} \dfrac{1-Q(x)}{2} L(u)(x) & \text{if } x \in \operatorname{int}(K); \\ 0 & \text{if } x \in \partial K \end{cases}$$

$(u \in C^2(K), x \in K)$.
Moreover, for every $m \geq 1$, T_L maps $P_m(K)$ into $P_m(K)$.

In particular, if K is the closed ball (with respect to the Euclidean norm $\|\cdot\|_2$) with center $\overline{x} \in \mathbf{R}^d$ and radius $r > 0$ and if $L = \Delta$, then

$$W_\Delta(u)(x) = \begin{cases} \dfrac{r^2 - \|x - \overline{x}\|_2^2}{2d} \Delta(u)(x) & \text{if } \|x - \overline{x}\|_2 < r; \\ 0 & \text{if } \|x - \overline{x}\|_2 = r \end{cases} \tag{1.39}$$

$(u \in C^2(K), x \in K)$ and T_Δ maps $P_m(K)$ into $P_m(K)$ for every $m \geq 1$.

5. Markov operators preserving polynomials

The main assumption in Theorem 2.5 involves the invariance under T of the spaces of polynomials of degree m, $m \geq 1$. Such a property, that seems to have its own independent interest, will be discussed below in more details.

As a first simple remark, note that, if T satisfies (1.25), then for every $\lambda \in [0,1]$ the operator $U_\lambda := \lambda T + (1-\lambda)I$ satisfies the same property.

We begin by presenting a counterexample to (1.25).

Example

Let $K = K_2$ be the canonical simplex of $\mathbf{R}^2$ and consider the Poisson operator $T_\Delta : C(K_2) \longrightarrow C(K_2)$ associated with the Laplace operator

$$\Delta u(x,y) := \frac{\partial^2 u}{\partial x^2}(x,y) + \frac{\partial^2 u}{\partial y^2}(x,y)$$

$(u \in C^2(int(K_2)), (x,y) \in int(K_2))$. Then

$$T_\Delta(P_2(K_2)) \not\subset P_2(K_2).$$

Below we shall consider another property similar to (1.25), namely

$$T(P_2(K)) \subset P_1(K), \tag{1.40}$$

i.e.,

$$T(h_1 h_2) \in P_1(K) \text{ for every } h_1, h_2 \in P_1(K).$$

Note that assumption (1.40) is satisfied when K is a simplex and T is the canonical projection on $C(K)$.
In fact this is the only case where (1.40) can occur.

Theorem 3.1

Assume that there exists a Markov operator T on $C(K)$ satisfying (1.1) and (1.40). Then

K is a simplex and T is the canonical projection associated with it.

In particular, $T(P_m(K)) \subset P_1(K)$ for every $m \geq 2$.

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Assume that there exists a Markov operator T on $C(K)$ satisfying (1.1) and (1.40). Then

$$K \text{ is a simplex and } T \text{ is the canonical projection associated with it.}$$

$$\text{In particular, } T(P_m(K)) \subset P_1(K) \text{ for every } m \geq 2.$$

From Theorem 2.6 it follows that, if K is an ellipsoid, then several classes of Poisson operators associated with strictly elliptic operators verify (1.25).
The next result shows that the inclusion

$$T(P_2(K)) \subset P_2(K)$$

characterizes the ellipsoids between those convex compact subsets of $\mathbf{R}^d$ that are **strictly convex**, i.e.,

$$\partial_e K = \partial K.$$

In such a case, necessarily $\mathrm{int}(K) \neq \emptyset$ unless K is *trivial*, i.e., K reduces to a singleton.

Theorem 3.2

Given a non-trivial strictly convex compact subset K of $\mathbf{R}^d$, $d \geq 2$, the following statements are equivalent:

(i) There exists a non-trivial Markov operator T on $C(K)$, i.e., $T \neq I$, satisfying

$$T(h) = h \qquad \text{for every } h \in \{\mathbf{1}, pr_1, \ldots, pr_d\}, \tag{1.41}$$

and

$$T(P_m(K)) \subset P_m(K) \qquad \text{for every } m \geq 2. \tag{1.42}$$

(ii) There exists a non-trivial Markov operator T on $C(K)$ satisfying (1.41) such that

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(iii) ∂K is an ellipsoid defined by a quadratic form

$$Q(x-\overline{x}) := \sum_{i,j=1}^{d} r_{ij}(x_i-\overline{x}_i)(x_j-\overline{x}_j) \ (x=(x_i)_{1\le i\le d}\in \mathbf{R}^d) \text{ with}$$

center $\overline{x}=(\overline{x}_i)_{1\le i\le d}\in \mathbf{R}^d$.

Moreover, if T is a non-trivial Markov **projection** on $C(K)$ satisfying (1.41) and (1.43), then one and only one of the following statements holds true:

(a) For every $x\in int(K)$ the support $Supp(\tilde{\mu}_x^T)$ is contained in an affine hyperplane R_x through x and hence, for every $f\in C(K)$,

$$T(f)(x)=\int_{\partial K\cap R_x} f\, d\tilde{\mu}_x^T. \qquad (1.44)$$

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Moreover, if T is a non-trivial Markov **projection** on $C(K)$ satisfying (1.41) and (1.43), then one and only one of the following statements holds true:

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$$T(f)(x) = \int_{\partial K\cap R_x} f\, d\tilde{\mu}_x^T. \qquad (1.44)$$

Theorem 3.2

(b) T is the Poisson operator associated with a suitable strictly elliptic differential operator of the form

$$L(u) := \sum_{i,j=1}^{d} c_{ij} \frac{\partial^2 u}{\partial x_i \partial x_j}$$

whose coefficients $(c_{ij})_{1 \le i,j \le d}$ are constant and satisfy

$$\sum_{i,j=1}^{d} r_{ij} c_{ij} = 1.$$

In the paper

- **F. Altomare and I. Raşa,**
  Towards a characterization of a class of differential operators associated with positive projections,
  Atti Sem. Mat. Fis. Univ. Modena, Supplemento al n. XLVI, **1998**, 3 - 38.

the reader can find a complete description of those convex compact subsets K of $\mathbf{R}^2$ such that there exists a **Markov projection** T on $C(K)$ satisfying (1.41) and (1.42).

In higher dimension we have no so complete results. However, below we mention two particular cases where properties (1.41) and (1.42) are reproduced.

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Tensorial products

Consider a finite family $(K_i)_{1 \leq i \leq d}$ of convex compact subsets having non-empty interior, each contained in some $\mathbf{R}^{s_i}$, $s_i \geq 1$, $i = 1, \ldots, d$. For every $i = 1, \ldots, d$, let $T_i : C(K_i) \longrightarrow C(K_i)$ be a Markov operator satisfying (1.41) and (1.42).

Setting

$$K := \prod_{i=1}^{d} K_i$$

and denoting by

$$T := \bigotimes_{i=1}^{d} T_i$$

the *tensor product* of $(T_i)_{1 \leq i \leq d}$, then T **is a Markov operator on** $C(K)$ **which satisfies (1.41) and (1.42)**.

In such a case it is also possible to describe the relevant differential operator.

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In such a case it is also possible to describe the relevant differential operator.

For the sake of simplicity we describe the simple case where

$$K_i = [0,1] \text{ for every } i = 1, \dots, d.$$

Let $Q_d := [0,1]^d$, $d \geq 1$, and for every $i = 1, \dots, d$ consider a Markov operator T_i on $C([0,1])$ satisfying (1.41) and (1.42).

If $T := \bigotimes_{i=1}^{d} T_i : C(Q_d) \to C(Q_d)$, then, for every $u \in C^2(Q_d)$ and $x = (x_i)_{1 \leq i \leq d} \in Q_d$,

$$W_T(u)(x) = \frac{1}{2} \sum_{i=1}^{d} \alpha_i(x) \frac{\partial^2 u}{\partial x_i^2}(x), \qquad (1.45)$$

where $\alpha_i(x) := T_i(e_2)(x_i) - x_i^2$ $(1 \leq i \leq d)$.

Finally note that, if $T_i = T_1$ for any $i = 1, \dots, d$, then

$$W_T(u)(x) = \frac{1}{2} \sum_{i=1}^{d} x_i(1 - x_i) \frac{\partial^2 u}{\partial x_i^2}(x) \qquad (1.46)$$

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We finally mention that, if S and T are two Markov operators on $C(K)$ satisfying (1.41) and (1.42), then the same properties are satisfied by the Markov operator

$$Z := \frac{S+T}{2}$$

From Theorem (2.5) it turns out that

$$W_Z = \frac{W_S + W_T}{4}$$

and hence the sum

$$W_S + W_T = 4W_Z,$$

defined on $C^2(K)$, is closable and its closure generates a Markov semigroup $(T(t))_{t\geq 0}$, which is the rescaled semigroup with parameter 4 of the semigroup generated by the closure of $(W_Z, C^2(K))$.
This result is not trivial because, in general, as it is well-known, the investigation of the generation property of the sum of two generators is a delicate problem.

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Thank you for your attention