Positive solutions of evolution equations governed by forms

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Bounded Operators

Let E be a Banach lattice.

$A \in \mathcal{L}(E)$.

If $A \geq 0$, then $e^{tA} := \sum_{k=0}^{\infty} \frac{t^k}{k!} A^k \geq 0$.

If $A + cI \geq 0$, then $e^{tc} e^{tA} = e^{t(c+A)} \geq 0$.
Thus $e^{tA} \geq 0$.

Theorem

Equivalent are

1. $e^{-tA} \geq 0 \quad \forall t \geq 0$
2. $-A + cI \geq 0$ for some $c \geq 0$
3. $(Au^+ | Au^-) \leq 0 \quad \forall u \in E$
   if $E = L^2(\Omega)$.

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The Lumer-Phillips Theorem

Let H be a Hilbert space, A an operator on H, i.e. $A\colon D(A) \to H$, $D(A)$ a subspace of H.

Theorem

Equivalent are

1. $-A$ generates a contraction semigroup.
2.
   - $\lambda + A\colon D(A) \to H$ is surjective for some $\lambda > 0$ and
   - $(Av \,|\, v) \geq 0 \quad \forall v \in D(A)$.

Then $(\lambda + A)^{-1}$ exists for $\lambda > 0$, $\|\lambda(\lambda + A)^{-1}\| \leq 1$ and

$$e^{-tA}v = \lim_{n\to\infty}(I + \frac{t}{n}A)^{-n}v, \; v \in H$$

defines the semigroup generated by $-A$.

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Positive Contraction semigroups

Let $\Omega \subset \mathbb{R}^d$ be open, $H = L^2(\Omega)$.
Let A be an operator on $L^2(\Omega)$.

Theorem (R. Phillips)

Equivalent are

1. $-A$ generates a positive contraction semigroup
2. - $\exists \lambda \geq 0$ such that $(\lambda + A)D(A) = H$ and
   - $(Au \mid u^+) \geq 0 \quad \forall u \in D(A)$.

Proof.

Show $(\lambda + A)^{-1} \geq 0$.
Let $\lambda u + Au \leq 0$. Show that $u \leq 0$.

$$0 \geq (\lambda u + Au \mid u^+) = \lambda \|u^+\|^2 + (Au \mid u^+) \geq \lambda \|u^+\|^2.$$

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Forms

Let $V \hookrightarrow_d H$, $\mathfrak{a}\colon V \times V \to \mathbb{R}$ bilinear with

$$|\mathfrak{a}(u,v)| \le M\|u\|_V\|v\|_V \qquad \text{(continuity)}$$
$$\mathfrak{a}(u,u) \ge \alpha\|u\|_V^2 \qquad \text{(coerciveness)}$$

Define $A \sim \mathfrak{a}$ on H by

$$D(A) = \{u \in V : \ \exists f \in H, \mathfrak{a}(u,v) = (f \mid v)_H \ \ \forall v \in V\}$$
$$Au = f$$

Thus $(Au \mid v) = \mathfrak{a}(u,v)$ for all $v \in V$.

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Invariant sets

Let $C \subset H$ be convex, closed and P the orthogonal projection on C.

Theorem (E. Ouhabaz)

Equivalent are:

1. $e^{-tA}C \subset C \ \forall t > 0$.
2. $v \in V \Rightarrow Pv \in C$ and $\mathfrak{a}(Pv, v - Pv) \geq 0$.

Beurling-Deny

Let $H = L^2(\Omega)$, $C = L^2(\Omega)_+$.

Then $Pv = v^+$ and $v - Pv = -v^-$.

Corollary (Beurling-Deny)

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Dirichlet-Laplacian

$\Omega \subset \mathbb{R}^d$ open, $H = L^2(\Omega)$.

$$H^1(\Omega) := \{v \in L^2(\Omega) : \; D_j v \in L^2(\Omega), j = 1, \ldots, d\}$$

$$V := H_0^1(\Omega) := \overline{\mathcal{D}(\Omega)}^{H^1}, \;\; \mathcal{D}(\Omega) := C_c^\infty(\Omega).$$

$$\mathfrak{a}(u, v) := \sum_{j=1}^d \int_\Omega D_j u D_j v.$$

$u \in H_0^1(\Omega) \;\Rightarrow\; u^+ \in H_0^1(\Omega)$ & $D_j u^+ = 1_{\{u>0\}} D_j u$.

Thus $\mathfrak{a}(u^+, u^-) = 0$ ($\mathfrak{a}$ is a *local* form)

Let $\mathfrak{a} \sim A$. Then

$$D(A) = \{v \in H_0^1(\Omega) : \; \Delta v \in L^2(\Omega)\}, \quad Av = \Delta v.$$

Set $\Delta^D := -A$. Thus $e^{t\Delta^D} \geq 0$.

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Irreducibility

Assume $e^{-tA} \geq 0$.

$(e^{-tA})_{t\geq 0}$ is called *irreducible*, if each closed invariant ideal $J \subset L^2(\Omega)$ is trivial.

$$\Leftrightarrow (f > 0 \Rightarrow (e^{-tA} f) \gg 0)$$

$$J = L^2(Y) := \{f \in L^2(\Omega) : \ f = 0 \text{ on } \Omega \setminus Y\}$$
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Criterion for irreducibility

Corollary

Assume that $\mathfrak{a}$ is local. Equivalent are:

1. $(e^{tA})_{t\geq 0}$ is irreducible
2. $Y \subset \Omega$ measurable, $1_Y V \subset V$
   $\Rightarrow |Y| = 0$ or $|\Omega \setminus Y| = 0$.

Example

$(e^{t\Delta^D})_{t\geq 0}$ is irreducible.

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Open problem

Consider the Laplacian Δ_1 on $L^1(\mathbb{R}^d)$, $d \geq 2$; i.e.

$$D(\Delta_1) := \{v \in L^1(\mathbb{R}^d) : \ \Delta v \in L^1(\mathbb{R}^d)\}$$
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Problem (Bénilan-Brézis)

Is Δ_1 local?

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$v \in D(\Delta_1) \not\Rightarrow D_1^2 u \in L^1_{\text{loc}}(\mathbb{R}^d)$

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Orthomorphisms

Let E be a Banach lattice.

$$\begin{aligned} A\colon E \to E \ \textit{local} :&\Leftrightarrow Au \in \{u\}^{dd} \\ &\Leftrightarrow A \ \textit{orthomorphism} \end{aligned}$$

Theorem (B. de Pagter)

A local $\Rightarrow$ A bounded.

Theorem (A. Zaanen 1975)

$E = L^p(\Omega)$, $1 \leq p \leq \infty$.
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Let $H = L^2(\Omega)$, $V \underset{d}{\hookrightarrow} H$, $a\colon V \times V \to \mathbb{R}$ continuous, coercive and $A \sim \mathfrak{a}$.

Theorem (W.A., S. Thomaschewski)

Equivalent are:

1. V is a sublattice of $L^2(\Omega)$, $\mathfrak{a}(u^+, u^-) = 0$ for all $u \in V$ and the cone of V is normal;
2. V is an ideal of $L^2(\Omega)$ and $\mathfrak{a}(u^+, u^-) = 0$;
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Non-autonomous forms

Let $V \hookrightarrow_d H$ and $a\colon [0,\tau] \times V \times V \to \mathbb{R}$ with

- $|\mathfrak{a}(t,u,v)| \le M\|u\|_V\|v\|_V$
- $\mathfrak{a}(t,u,u) \ge \alpha\|u\|_V^2$
- $\mathfrak{a}(\cdot,u,v)\colon [0,\tau] \to \mathbb{R}$ Lipschitz
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$A(t) \sim \mathfrak{a}(t,\cdot,\cdot)$

$$H^1(0,\tau;H) := \{u \in C([0,\tau];H) : \exists \dot{u} \in L^2(0,\tau;H),$$
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Maximal regularity

Theorem (W.A., D. Dier, H. Laari, E. Ouhabaz 2012)

Let $u_0 \in V$, $f \in L^2(0, \tau; H)$.
Then $\exists! u \in H^1(0, \tau; H)$ s.t. $u(t) \in D(A(t))$ a.e. and

$$\begin{aligned} \dot{u}(t) + A(t)u(t) &= f(t) \text{ a.e.} \\ u(0) &= u_0. \end{aligned}$$

Moreover, $u \in C([0, \tau]; V)$!

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Let $C \subset H$ be closed, convex and $P\colon H \to C$ the orthogonal projection on C.

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Assume

1. $v \in V \Rightarrow Pv \in V$;
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Non-autonomous Robin boundary conditions

Let $\Omega \subset \mathbb{R}^d$ be a bounded open set with Lipschitz boundary $\Gamma := \partial\Omega$. Let $\beta\colon [0,\tau] \to L^\infty(\Gamma)$ be Lipschitz.

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Positivity: If $f(t) \geq 0$ and $u_0 \geq 0$, then $u(t) \geq 0$ for all $t \in [0,\tau]$.

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Leiden 1938
Advisor: Johannes Droste

"Concerning eigenfunction expansions of certain boundary value problems"