# $\Lambda(p)$–SPACES

Sergey Astashkin, Samara State University

**Definition 1.** We say that a closed subspace $H$ of the space $L_p = L_p[0,1]$, $1 \le p < \infty$, is a $\Lambda(p)$-*space* if from $f_n \in H$ $(n = 1, 2, \dots)$ and $f_n \to 0$ in measure it follows $\|f_n\|_p \to 0$. Equivalently, for each (for some) $0 < q < p$ there exists a constant $C_q > 0$ such that

$$\|f\|_p \le C_q \|f\|_q \quad \text{for all} \quad f \in H \tag{1}$$

**Examples:**

(a) Khintchine inequality: for every $0 < p < \infty$, there exist $A_p, B_p > 0$ such that for any $\{c_k\}_{k=1}^{\infty} \in l_2$ we have

$$A_p \|(c_k)\|_2 \le \Big\| \sum_{k=1}^{\infty} c_k r_k \Big\|_p \le B_p \|(c_k)\|_2. \tag{2}$$

Here,

$$\|(c_k)\|_2 := \left( \sum_{k=1}^{\infty} c_k^2 \right)^{1/2},$$

$$r_k(t) = \operatorname{sign}(\sin 2^k \pi t), \;\; k \in \mathbb{N}, t \in [0,1].$$

By (2), for every $1 \le p < \infty$ $\{r_k\}$ is equivalent in $L_p$ to the standard basis of $l_2$ and $[r_k]$ is a $\Lambda(p)$-space.

(b) Let $\{\xi_k\}_{k=1}^{\infty}$ be a sequence of independent normalized $s$-stable random variables $(1 < s < 2)$. This means that

$$\int_{\mathbb{R}} e^{it\xi_k(x)}\, dx = e^{-|t|^s}. \quad k = 1, 2, \ldots$$

Then $[\xi_k]$ is a $\Lambda(p)$-space iff $p \in (0, s)$ (and also $[\xi_k] \approx l_s$).

**W. Rudin, "Triginometric series with gaps", 1960**:

For a given set $E \subset \mathbb{Z}$ define the subspace $H_E = [e^{ikt}]_{k\in E}$ of $L_p[0, 2\pi)$, $0 < p < \infty$. A set $E$ is called a $\Lambda(p)$-*set* if $H_E$ is a $\Lambda(p)$-space or, equivalently, if for some $0 < q < p$ there exists a constant $C_q > 0$ such that inequality (1) holds for every polynomial $f$ with spectrum (i.e., support of Fourier transform) in $E$. For all integers $n > 1$ there exist $\Lambda(2n)$-sets that are not $\Lambda(q)$-sets for every $q > 2n$.

**G.F. Bachelis and S.E. Ebenstein, 1974**:

If $p \in (1, 2)$, then every $\Lambda(p)$-set is a $\Lambda(q)$-set for some $q > p$.

**J. Bourgain, 1989**:

For all $p > 2$ there exist $\Lambda(p)$-sets that are not $\Lambda(q)$-sets for every $q > p$.

These classical results show a critical difference between the cases $1 < p < 2$ and $2 < p < \infty$.

**$\Lambda(p)$-spaces $H$ whose unit ball $B_H := \{f \in H : \|f\|_p \leq 1\}$ has equi-absolutely continuous norms in $L_p$.**

**Definition 2.** A set $K \subset L_p$ has *equi-absolutely continuous norms* in $L_p$ if

$$\lim_{\delta \to 0} \sup_{m(E)<\delta} \sup_{f \in K} \|f\chi_E\|_p = 0.$$

Every closed subspace $H$ of $L_p$ whose unit ball $B_H$ has equi-absolutely continuous norms in $L_p$ is a $\Lambda(p)$-space.

A version of the De La Vallee Poussin's theorem on uniformly integrable sets in $L_1$:

**Theorem 1.** *Let $1 \leq p < \infty$, and let $H$ be a closed subspace of $L_p$. The following conditions are equivalent:*

*(i) the unit ball of $H$ has equi-absolutely continuous norms in $L_p$;*

*(ii) there is an Orlicz function $N$ such that $H \subset L_N \subset L_p$ and*

$$\lim_{u \to \infty} N(u)u^{-p} = \infty.$$

*Example* 1. Let $1 \leq p < \infty$, and let $f_0$ be a mean zero function independent of $\{r_k\}_{k=1}^{\infty}$ and such that $f_0 \in L_p \setminus \cup_{q>p} L_q$. Then, the unit ball of the subspace $H$ spanned by the set $\{f_0\} \cup \{r_k\}_{k=1}^{\infty}$ has equi-absolutely continuous norms in $L_p$. But $H \not\subset L_q$ and therefore it is not a $\Lambda(q)$-space for any $q > p$.

**When the unit ball $B_H := \{f \in H : \|f\|_p \leq 1\}$ of a $\Lambda(p)$-space $H$ has equi-absolutely continuous norms in $L_p$?**

The answer essentially depends on the value of $p$.

(i) $1 \leq p < 2$:

**Theorem 2.** *If $H$ is a $\Lambda(p)$-space, where $1 \leq p < 2$, then its unit ball $B_H$ has equi-absolutely continuous norms in $L_p$.*

The situation is completely different in the case when $p \geq 2$.

1. For every $p > 2$ there is a $\Lambda(p)$-space $H_p$ such that $H_p \not\subset L_N$ whenever $L_N \overset{\neq}{\subset} L_p$.

J. Bourgain, 1989.

2. There is a $\Lambda(2)$-space $H_2$ spanned by a sequence of mean zero independent functions such that $H_2 \not\subset X$ for any r.i. space $X$ such that $X \overset{\neq}{\subset} L_2$.

Therefore, by Theorem 1, for every $p \geq 2$ there exists a $\Lambda(p)$-space whose unit ball fails to have equi-absolutely continuous norms in $L_p$.

At the same time, if a $\Lambda(2)$-space is obtained by a special way, its unit ball has equi-absolutely continuous norms in $L_2$.

**Definition 3.** A Banach space $X$ of real-valued Lebesgue–measurable functions on the interval $[0,1]$ is called *rearrangement invariant (r.i.)* if 1) from $y \in X$ and $|x(t)| \le |y(t)|$ a.e. on $[0,1]$ it follows that $x \in X$ and $\|x\|_X \le \|y\|_X$; 2) from $y \in X$ and

$$m\{s \in [0,1] : |x(s)| > \tau\}$$

$$= m\{s \in [0,1] : |y(s)| > \tau\} \quad (t > 0).$$

it follows that $x \in X$ and $\|x\|_X = \|y\|_X$ ($m$ is the Lebesgue measure).

**Definition 4.** Let $N(u)$ be an Orlicz function on $[0,\infty)$, that is, a continuous convex increasing function on $[0,\infty)$ such that $N(0) = 0$. Then the *Orlicz space* $L_N$ consists of all measurable functions $x(t)$ on $[0,1]$ such that $\int_0^1 N(|x(t)|/\lambda)\, dt \le 1$, for some $\lambda > 0$. The *Luxemburg norm* $\|x\|_{L_N} := \inf \lambda$, where the infimum is taken over all $\lambda$ satisfying the last inequality.

**Definition 5.** A linear operator between two Banach spaces $E$ and $F$ is called *strictly singular* (SS) if there is no an infinite dimensional subspace $H$ of $E$ such that the restriction $T|_H$ is an isomorphism (T. Kato). If $E$ is a Banach lattice and we consider only subspaces $[x_n]$ spanned by disjoint sequences of non-null vectors $\{x_n\}_{n=1}^{\infty}$ in $E$, a bounded operator $T : E \to F$ is said to be *disjointly strictly singular* (DSS) (F.L. Hernandez and B. Rodrigues-Salinas).

Every SS operator is DSS but the converse is not true in general; the canonic inclusion $L_q[0,1] \to L_p[0,1]$, $1 \leq p < q < \infty$, is DSS but is not SS (consider Rademacher functions).

Some of $\Lambda(p)$-spaces are generated by DSS-inclusions of r.i. spaces into $L_p$ :

**Theorem 3.** *Let $1 \leq p < \infty$, and let $H$ be a closed subspace of $L_p$. Suppose that there is an r.i. space $X$ such that $H \subset X \subset L_p$ and the inclusion operator $I : X \to L_p$ is DSS. Then $H$ is a $\Lambda(p)$-space.*

(ii) $p = 2$ :

**Theorem 4.** *Let $X$ be an r.i. space on $[0,1]$ such that $X \subset L_2$, and let the norms of $L_2$ and $X$ be equivalent on some closed subspace $H$ of $L_2$. Then, if the inclusion operator $I : X \to L_2$ is DSS, the unit ball $B_H$ of $H$ has equi-absolutely continuous norms in $L_2$.*

*Sketch of the proof of Theorems 2 and 4:*

Let $1 < p \leq 2$. Assume that the unit ball $B_H$ of a $\Lambda(p)$-space $H$ fails to have equi-absolutely continuous norms in $L_p$. Then, we can construct a bounded unconditional basic sequence $\{f_n\}_{n=1}^{\infty}$ in $L_p$, $f_n \in H$ $(n = 1, 2, \dots)$, such that

$$\|f_n \chi_{F_n}\|_p \geq \alpha \quad (n = 1, 2, \dots). \tag{3}$$

for some $\alpha > 0$ and pairwise disjoint sets $F_n \subset [0,1]$. From unconditionality of $\{f_n\}$ and inequalities (3) we deduce that the sequence $\{f_n\}$ is equivalent to the standard basis of $l_p$. Now, if $1 < p < 2$, by Rosenthal's theorem (1973), we conclude that $H$ fails to be a $\Lambda(p)$-space, and this contradiction finishes the proof in this case. If $p = 2$, we will need the extra condition that there is an r.i. space $X$ such that $X \subset L_2$, the norms of $L_2$ and $X$ are equivalent on $H$ and the inclusion operator $I : X \to L_2$ is DSS. Then, we prove that the norms of $X$ and $L_2$ are equivalent on the subspace $[f_n \chi_{F_n}]_{n=1}^{\infty}$, and again come to contradiction.

(iii) $p > 2$:

**Theorem 5.** *For arbitrary $p > 2$ there exists an Orlicz space $L_F$ with the following properties: $L_F \subset L_p$, the inclusion operator $I : L_F \to L_p$ is DSS, and the norms of $L_p$ and $L_F$ are equivalent on some $\Lambda(p)$-space $H$ whose unit ball $B_H$ fails to have equi-absolutely continuous norms in $L_p$.*

*Sketch of the proof.* Firstly, using some constructions from the paper of F.L. Hernandez and B. Rodriguez-Salinas (1989), we can find an Orlicz function $F$ such that $L_F \subset L_p$, the inclusion operator $I : L_F \to L_p$ is DSS, and

$$F(a_n) \leq C_q a_n^p \;\; (n \in \mathbb{N}), \tag{4}$$

for some sequence $a_n \to +\infty$ and constant $C_q > 0$.

We set $b_n := 1/F(a_n)$, $n = 1, 2, \ldots$ Then, the subspace $H$ spanned by a sequence of independent functions $\{f_k\}_{k=1}^{\infty}$ on $[0,1]$ such that $\int_0^1 f_k(t)\,dt = 0$, $|f_k(t)| = b_k^{-1/p}$ if $t \in E_k$, where $m(E_k) = b_k$, and $|f_k(t)| = 1$ if $t \in [0,1] \setminus E_k$ ($k = 1, 2, \ldots$), satisfies all required conditions. Indeed, using (4), independence of $f_k$, and some properties of $F$, we prove that

$$\Big\| \sum_{k=1}^{\infty} c_k f_k \Big\|_p \asymp \Big\| \sum_{k=1}^{\infty} c_k f_k \Big\|_{L_F} \asymp \|(c_k)\|_2. \tag{5}$$

From (5), DSS-property of the inclusion $L_F \subset L_p$, and Theorem 3 it follows that the subspace $H = [f_k]$

in $L_p$ is a $\Lambda(p)$-space. On the other hand, since

$$\|f_k \chi_{E_k}\|_p = 1 \;\; (k = 1, 2, \dots)$$

and

$$m(E_k) = b_k \to 0 \;\text{ as }\; k \to \infty,$$

the unit ball $B_H$ fails to have equi-absolutely continuous norms in $L_p$.

**R. Blei, "Analysis in Integer and Fractional Dimensions", 2001**:

A closed subspace $H$ of $L_2$ is a $\Lambda(2)$-space iff there exists $\delta > 0$ such that for every $f \in H$ there is a function $g \in H^\perp$, with $f + g \in L_\infty$ and $\|f + g\|_\infty \leq \delta \|f\|_2$ ($H^\perp$ is the orthogonal complement of $H$).

**Definition 6.** A closed subspace $H$ of $L_2$ is called a *uniformizable* $\Lambda(2)$-*space* if for every $\varepsilon > 0$ there exists $\delta > 0$ such that for arbitrary $f \in H$ there is $g \in H^\perp$ satisfying the conditions $f + g \in L_\infty$, $\|f + g\|_\infty \leq \delta \|f\|_2$, and $\|g\|_2 \leq \varepsilon \|f\|_2$.

**Theorem 6.** *A* $\Lambda(2)$-*space* $H$ *is uniformizable iff the unit ball* $B_H$ *of* $H$ *has equi-absolutely continuous norms in* $L_2$.

"$\Lambda(2)$-set union problem" (R. Blei, 2001): Let $H_1$ and $H_2$ be mutually orthogonal infinite-dimensional $\Lambda(2)$-spaces. Is their direct sum $H_1 \oplus H_2$ a $\Lambda(2)$-space?

**Theorem 7.** *There exist mutually orthogonal* $\Lambda(2)$-*spaces* $H_1$ *and* $H_2$ *such that the sum* $H_1 \oplus H_2$ *is not a* $\Lambda(2)$-*space.*

If at least one of $\Lambda(2)$-spaces is uniformizable, their sum is a $\Lambda(2)$-space as well.