About comonotonicity and the Choquet integral

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Outline

Definition

Two real valued functions f and g, defined on a set X, are said to be comonotonic if:

$$(f(x) - f(y))(g(x) - g(y)) \geq 0$$

for all $x, y \in X$.

Definition

Let Ω be an abstract set and $\mathcal{F}$ be a family of subsets of Ω containing $\emptyset$ and Ω. A Choquet capacity is a map γ, from $\mathcal{F}$ to R^+, such that:

$$\gamma(\emptyset) = 0 \text{ and } \gamma(A) \leq \gamma(B) \text{ if } A \subset B$$

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Definition

Let $f \geq 0$ be a function on Ω. The Choquet integral is defined by:

$$\gamma(f) = \int_0^{\infty} \gamma(f \geq t)dt$$

In the general case:

$$\gamma(f) = \int_0^{\infty} \gamma(f \geq t) + \int_{-\infty}^{0} [\gamma(f \geq t) - \gamma(\Omega)]dt$$

Theorem

The Choquet integral is additive on comonotic pairs.

S is the set of the states of nature. C is the set of consequences.
F is the set of acts, or decisions, i.e. the set of maps from S to C.
Example: you have a house in good condition. After one year:
s_1: the house is in good condition.
s_2: the house is destroyed.
You have to choose an insurance d; the premium is π_d:
s_1: after one year, you have: $L - \pi_d$. (L: price of the house).
s_2: after one year, you have: L_d. (you get L_d).
You have to compare the various possible d.
Two cases are in order:
1) Decision under risk: You knows p the probability of s_1.
You have to compare the various probabilites on R:

$$P_d = p\varepsilon_{(L-\pi_d)} + (1-p)\varepsilon_{L_d}$$

2) Decision under uncertainty.

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Decision under risk.

We deal with lotteries, i.e: discrete probabilities P on an abstract set C.

$P = (x_1, p_1; \ldots; x_i, p_i; \ldots; x_n, p_n)$, where $x_i \in C$ and $\sum p_i = 1$.

We assume:

1) A total preorder $\geq$ on the set $\mathcal{L}_0$ of lotteries on C.

2) Continuity: If $P > Q > R$, there are $a, b \in]0, 1[$ with:

$$aP + (1-a)R > Q > bP + (1-b)R$$

3) Independence: If $P \geq Q$, for any R and $0 < a < 1$, then:

$$aP + (1-a)R \geq aQ + (1-a)R$$

Theorem (von Neumann, Morgenstern)

There is a utility functions u from C to R such that:

$$(P \geq Q) \text{ iff } (P(u) \geq Q(u))$$

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The Allais paradox (example due to Kahneman and Tversky):

1)

A: You get 3.000 Euros with proba 1.

B: You get 4.000 Euros with proba 0, 8 or 0 with proba 0, 2.

A lot a people prefer A to B.

2)

C: You get 3.000 Euros with proba 0, 25 or 0 with proba 0, 75.

D: You get 4.000 Euros with proba 0, 2 or 0 with proba 0, 8.

A lot of people prefer D to C.

However:

$$P_C = 0,25 P_A + 0,75 \varepsilon_0 \text{ and } P_D = 0,25 P_B + 0,75 \varepsilon_0$$

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D: You get 4.000 Euros with proba $0,2$ or 0 with proba $0,8$.
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Alain Chateauneuf (1999) has proposed a weaker axiomatic:
C is a connected, compact, metric space. $\mathcal{L}_0$ is equipped with a total preorder $\geq$ (note that $\geq$ induced a total preorder on C) such that:
1) Continuity.
2) Monotonicity: $P \geq_D Q$ then $P \geq Q$.
3) Comonotonic sure-thing principle (C.S.T.P.):
Let $P = \sum_1^n p_i \varepsilon_{x_i}$ and $Q = \sum_1^n p_i \varepsilon_{y_i}$, written in rank order, with $P \geq Q$. If $x_{i_0} = y_{i_0}$ for some $1 \leq i_0 \leq n$ and if we replace $x_{i_0} = y_{i_0}$ by the same element $x'_{i_0} = y'_{i_0}$ in C, to get P' and Q', so that x'_{i_0} and y'_{i_0} has the same rank i_0, then $P' \geq Q'$.
4) Comonotonic mixture independence axiom (C.M.I.A.)

Theorem (A. Chateauneuf)

There exists a continuous function u from C to R, and a strictly increasing continuous function f from $[0, 1]$ onto itself, such that:
If we set, for $P = \sum_1^n p_i \varepsilon_{x_i}$, $U(P) = \sum_{i=1}^n (f(\sum_i^n p_j) - f(\sum_{i+1}^n p_j))u(x_i)$, then $P \geq Q$ iff $U(P) \geq U(Q)$.

$U(P)$ is the rank-dependent expected utility of P (R.D.E.U.).

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We can assume $u \geq 0$. Then $U(P) = \sum_{i=1}^{n}(f(\sum_{i}^{n} p_j) - f(\sum_{i+1}^{n} p_j))u(x_i)$, can be written as a Choquet integral:
If we set: $\mu_P(E) = f(P(E))$ for $E \subset C$, we have:

$$U(P) = \mu_P(u)$$

Decision under uncertainty.

We deal within the framework of L. Savage.

S is equipped with a Boolean algebra $\mathcal{B}$.

We deal with the set F of acts f (maps from S to C) of the following form:

There exists a finite $\mathcal{B}$-measurable partition of S such that f is constant on each element of the partition.

L. Savage has introduced 7 postulates:

1) F is equipped with a total preorder $\geq$. (inducing a total preorder on C).

2) Let $f, g \in F$ be such that $f \geq g$ and $f = g$ on $E \in \mathcal{B}$. Then, for all $f', g' \in F$, with $f' = f$ on E^c, $g' = g$ on E^c, and $f' = g'$ on E, one has $f' \geq g'$.

This is called the Sure Thing Principle.

Theorem (L. Savage)

There exist a unique finitely additive probability measure π on $\mathcal{B}$, and a bounded function u from S to R such that, if $f, g \in F$:

$$(f \geq g) \text{ iff } \left(\int u(f(s))\pi(ds) \geq \int u(g(s))\pi(ds)\right)$$

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The Ellsberg paradox.
Suppose an urn contains 90 balls: 30 are Red, the others (60) are Blue or Yellow.
The set S is $\{R, B, Y\}$, with obvious notations. The set C is $\{0, 1\}$.
We consider the followings 4 acts:
$d_1 = 1(R)$
$d_2 = 1(B)$
$d_3 = 1(R \cup Y)$
$d_4 = 1(B \cup Y)$
A lot of people prefer d_1 to d_2 and d_4 to d_3.
But: $d_1 = d_3$ on $R \cup B$, $d_2 = d_4$ on $R \cup B$, $d_1 = d_2$ on Y, $d_3 = d_4$ on Y.
Whence a contradiction with the Sure Thing Principle.

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David Schmeidler has proposed a weaker axiomatic; here is a simplified version:
S is equipped with a σ algebra $\mathcal{A}$. The set set C is R. The set F of acts is the set of all bounded measurable functions from S to R.
There are 4 axioms:
1) A total preorder ($\geq$) on F.
2) Stability of $\geq$ under monotone uniform convergence of sequences.
3) Monotonicity: $X \geq Y + \varepsilon 1$ implies $X > Y$.
4) Comonotonic independance: If $X \geq Y$ and, if Z is comonotic with X and with Y, then $X + Z \geq Y + Z$.

Theorem (D. Schmeidler)

There exists a Choquet capacity γ on $\mathcal{A}$, with $\gamma(1) = 1$, such that, for $X, Y \in F$:

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Representation of comonotonic additive functionals

Recently, S. Cerreira-Vioglio, F. Maccheroni, M. Marinacci, and L. Montrucchio have established representations results for some classes of comonotonic additive functionals. They manage to encompass the following 2 settings:

1) When the functions space is the space of bounded measurable functions with respect to an algebra.

2) When the functions space is a Stone vector lattice of bounded functions.

Definition

A Stone lattice L is comonotonic if there exists a Stone vector lattice E such that $L \subset E$, and given any two comonotonic $f, g \in E$ and $\varepsilon > 0$, there are two comonotonic $f_\varepsilon, g_\varepsilon \in L$ with $\|f - f_\varepsilon\| \leq \varepsilon$, $\|g - g_\varepsilon\| \leq \varepsilon$, and $f_\varepsilon + g_\varepsilon \in L$.

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Variation of a functional

Definition

If E is a set of real valued functions and V is a real valued functional on E, for every $f \leq g \in E$ we set:

$$T_V(f,g) = \sup(\sum_1^n |V(f_i) - V(f_{i-1})|)$$

where the sup is taken over all finite chains in E:

$$f = f_0 \leq f_1 \leq \ldots \leq f_n = g$$

Theorem

Let L be a comonotonic Stone lattice of bounded functions and V be a comonotonic additive functional on L, which is of bounded variation and outer regular.
Then there exist two outer regular capacities ν_1, ν_2 on $\sum_L$ such that:

$$V(f) = \nu_1(f) - \nu_2(f) \text{ for } f \in L$$

Moreover, $\nu = \nu_1 - \nu_2$ is unique, as an outer continuous set function.

Theorem

Let L be a Stone vector lattice of bounded functions and V be a comonotonic additive functional on L, which is of bounded variation, pointwise continuous and superadditive.
Then there is a unique continuous and supermodular ν, of bounded variation, defined on the σ-algebra generated by L, such that:
$V(f) = \nu(f)$ *on L.*

Let E be a vector lattice of bounded functions on a set Ω, containing 1, and $\mathcal{A}$ be the boolean algebra of subsets of Ω generated by the sets $(f \geq 0)$, where $f \in E$.

If T is a positive linear form on E, is it possible to represent T by an integral with respect to an additive measure on $\mathcal{A}$?

Theorem (abstract Alexandroff Theorem)

The answer is yes whenever the space E is such that:
For every $f, g, h \in E^+$ with $g \leq f$, and $h(x) = 0$ whenever $f(x) = 0$, the function ϕ defined by:

$$\phi(x) = (g(x)/f(x))h(x) \text{ when } f(x) > 0 \text{ and } \phi(x) = 0 \text{ when } f(x) = 0$$

belongs to E.

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If $A = (f > 0)$ for some $f \in E$ we set: $\mu^*(A) = \sup\{T(g) : 0 \leq g \leq 1(A)\}$
If $B \subset \Omega$ we set $\mu^*(B) = \inf\{\mu^*(A)\}$ where $B \subset A$ and A is of the form $(f > 0)$.