Geometric characterization of algebra homomorphisms on f-algebras

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Leiden, July 25, 2013

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Birkhoff,1946

Extremal points of the convex set $\mathcal{B}$ are permutation matrices.

1961, A. Ionescu Tulcea and C. Ionescu Tulcea

Let X and Y are two compact Hausdorff spaces and the convex set

$$\mathfrak{K} = \{T \in \mathcal{L}(C(X), C(Y)), T \geq 0, \text{ and } T\mathbb{1} = \mathbb{1}\}$$

Then T is an algebra homomorphism in $\mathcal{L}(C(X), C(Y))$ if and only if T is an extreme point in $\mathfrak{K}$

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- The set of all orthomorphisms will be denoted $\mathrm{Orth}(A)$

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Huijsmans and de Pagter, 1984

if $T \in \mathcal{M}(A,B)$, then the following are equivalent.

(i) T is an extreme point in $\mathcal{M}(A,B)$.

(ii) T is an algebra homomorphism.

(iii) T is a Riesz homomorphism.

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Theorem

Let $T \in \mathcal{L}(A,B)^+$. Then T is an algebra homomorphism if and only if T is a lattice homomorphism and $T(u)$ is idempotent.

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Theorem

Let w be an idempotent element in B and $T \in \mathcal{K}_w(A,B)$. Then T is an algebra homomorphism if and only if T is an extreme point of $\mathcal{K}_w(A,B)$.

- For every positive element T in $\mathcal{L}(A,B)$, we put

$$\langle T\rangle = \{S \in \mathcal{L}(A,B) : -nT \leq S \leq nT \text{ for some } n = 1,2,...\}.$$

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Theorem

Let w be an idempotent element in B and $T \in \mathcal{K}_w(A,B)$. Then T is an algebra homomorphism if and only if $\widehat{u}$ is one-to-one.

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- A is an f-algebra with unit element $\longrightarrow A \simeq \mathrm{Orth}(A)$.

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Theorem

Let A be a Stone f-algebra with no unit elements. Then the following assertions hold.

(i) $A^{\triangleright}$ is a sub f-algebra of $\mathrm{Orth}\,(A)$.

(ii) A is a ring and order ideal in $A^{\triangleright}$.

Contractive operator

Let A and B be Stone f-algebra. An operator $T \in \mathcal{L}(A, B)$ is said to be contractive if

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Example

A be the set of all real-valued continuous functions f on $(0, \infty)$ for which there exist $r_f \in (0, \infty)$ and a real polynomial P_f such that

$$f(r) = P_f(r) \text{ for all } r \in (r_f, \infty).$$

If $B = \mathbf{R}$ and define $T \in \mathcal{L}(A, B)$ by

$$Tf = P_f(0) \text{ for all } f \in A.$$

-

Stone extension

Let A and B be f-algebra. Assume that A has no unit element. Then, any $T \in \mathcal{L}(A, B)$ has an obvious extension $T^{\triangleright} \in \mathcal{L}(A^{\triangleright}, B^{\triangleright})$, where

$$T^{\triangleright}(f + rI_A) = Tf + rI_B \text{ for all } f \in A \text{ and } r \in \mathbf{R}.$$

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Theorem

Let A be Stone f-algebra with no unit element and B be a Stone f-algebra. The following equivalences hold for any $T \in \mathcal{L}(A, B)$.

(i) $T \in \mathcal{K}(A, B)$ if and only if $T^{\rhd} \in \mathcal{K}(A^{\rhd}, B^{\rhd})$

(ii) T is an extreme point in $\mathcal{K}(A, B)$ if and only if $T^{\rhd}$ is an extreme point in $\mathcal{K}(A^{\rhd}, B^{\rhd})$.

(iii) T is an algebra hmomorphism in $\mathcal{K}(A, B)$ if and only if $T^{\rhd}$ is an algebra hmomorphism in $\mathcal{K}(A^{\rhd}, B^{\rhd})$.

Theorem

Let A and B be Stone f-algebras, and let $T \in \mathcal{K}(A, B)$. Then, T is an extreme point in $\mathcal{K}(A, B)$ if and only if T is an algebra homomorphism.

Stone Operator

Let A *and* B *be Stone* f*-algebras. An operator* $T \in \mathcal{L}(A, B)$ *is called a* Stone operator *if*

$$T(I_A \wedge f) = I_B \wedge Tf \text{ for all } f \in A.$$

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Stone Operator and Riesz homomorphism

Let A and B be Stone f-algebras, and $T \in \mathcal{K}(A, B)$. If T is a Stone operator, then T is a Riesz homomorphism.

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The converse need not to be true:

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T is Riesz homomorphism but not Stone homomorphism

Theorem

Let A and B be Stone f-algebras with no unit elements, and let $T \in \mathcal{K}(A, B)$. Then the following are equivalent.

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Theorem

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This talk is based among these articles:

- M.A. Ben Amor, K. Boulabiar, Almost f-maps and almost f-rings, *Algebra Univ.* **69** (2013) 93-99.
- M.A. Ben Amor, K. Boulabiar, A geometric characterization of ring homomorphisms on f-rings, *Journal of Algebra and its applications*, to appear.
- M.A. Ben Amor, K. Boulabiar, C. El Adeb ,Extreme contractive operators on Stone f-algebras, *preprint*
- C.B. Huijsmans and B. de Pagter, Subalgebras and Riesz subspaces of an f-algebra, *Proc. London Math. Soc.*, 48 (1984), 161-174.

Спасибо за внимание

Thank you for your attention