Isometries on $L^2(X)$ and monotone functions

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joint work with
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Índex

Preliminaries and Examples

- In [B-Soria,(2011)] optimal estimates concerning the norm of the Hardy averaging operator

$$Sf(x) = \frac{1}{x}\int_0^x f(t)\,dt,$$

minus the identity I, have been established on $L^p(\mathbb{R}^+)$ on the cone of decreasing functions, for the inequality:

$$\|Sf - f\|_{L^p(\mathbb{R}^+)} \leq C\|f\|_{L^p(\mathbb{R}^+)}.$$

- However, one of the most remarkable results is that $S - I$ is, in fact, an isometry on $L^2(\mathbb{R}^+)$

$$\|(S - I)f\|_{L^2(\mathbb{R}^+)} = \|f\|_{L^2(\mathbb{R}^+)}, \tag{1}$$

and the same holds for the adjoint operator S^*, defined by

$$S^*f(x) = \int_x^\infty \frac{f(t)}{t}\,dt,$$

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- The corresponding property fails for the discrete counterpart of the Hardy operator

$$S^d(\{a_n\}_{n\in\mathbb{N}})(m) = \frac{1}{m}\sum_{j=1}^{m} a_j.$$

- In [Brown, Halmos and Shields, (1965)], [Kaiblinger, Maligranda, Persson, (2000)] the isometry (1) was extended to the generalized Hardy operator acting on $L^2(w)$

$$S_w f(x) = \frac{1}{W(x)}\int_0^x f(t)w(t)\,dt,$$

where w is a weight on $\mathbb{R}^+$ and $W(x) = \int_0^x w(t)\,dt$.

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- What about this question in a more general setting (the discrete case, other kind of operators, etc.)?:

The isometric property on L^2 can be obtained just by looking at the action of the operator on certain classes of functions (e.g., in $\mathbb{R}^+$, to the cone of positive and decreasing functions).

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For this purpose, we will work on a measure space $(X, d\mu)$ and consider integral operators T_K bounded on $L^2(X)$ given by a kernel K defined on $X \times X$; that is,

$$T_K f(x) = \int_X K(x,y) f(y)\, d\mu(y), \quad f \in L^2(X),$$

under the assumption that its absolute value $|K(x,y)|$ (for simplicity we will take K to be real) defines also a bounded operator $T_{|K|}$ on $L^2(X)$.
Our aim is to study conditions on K so that

$$\|T_K f - f\|_{L^2(X)} = \|f\|_{L^2(X)}.$$

We will study also how the isometric property behaves when the operator T_K is projected to the subspace $L^2(Y)$, where $Y \subset X$. Brown et all considered both the Hardy operator and its adjoint, and the cases $X = \mathbb{R}^+$ and $Y = (0,1)$.

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Preliminaries and Examples

One classical result:

Lemma Let H be a real Hilbert space with the norm $\|x\| =< x, x >^{1/2}$. Let $T : H \longrightarrow H$ be a bounded operator and T^* its adjoint. Then,

(i) the operator T is an isometry; i.e., $\|Tx\| = \|x\|$ for every $x \in H$ if and only if $T^* \circ T = I$;

(ii) the operator T is a surjective isometry if and only if $T^* \circ T = T \circ T^* = I$.

Preliminaries and Examples

As a consequence, we get

Theorem Let T_K be an integral operator as before. Then, the following facts are equivalent:

(i) $T_K - I$ is an isometry on $L^2(X)$;

(ii) for almost all $(x, y) \in X \times X$,

$$\int_X K(z,x)K(z,y)\,d\mu(z) = K(x,y) + K(y,x).$$

Preliminaries and Examples

Again, we can formulate an analogue to the previous theorem which establishes necessary and sufficient conditions on K to obtain a surjective isometry:

Theorem The following facts are equivalent:

- **(i)** $T_K - I$ is a surjective isometry on $L^2(X)$;
- **(ii)** $T_K^* - I$ is a surjective isometry on $L^2(X)$;
- **(iii)** for almost all $(x, y) \in X \times X$,

$$\begin{aligned} \int_X K(z,x)K(z,y)\,d\mu(z) &= \int_X K(x,z)K(y,z)\,d\mu(z) \\ &= K(x,y) + K(y,x). \end{aligned}$$

Preliminaries and Examples

As a corollary, if the previous theorem is applied to the kernel

$$K(x,y) = \frac{1}{x}\chi_{(0,x)}(y)\frac{1}{w(y)},$$

with respect to the measure $d\mu(y) = w(y)\,dy$ on $\mathbb{R}^+$, we obtain the following.

Corollary Let S be the Hardy operator and w be a weight on $\mathbb{R}^+$. Then, $S - I$ is an isometry on $L^2(w)$ if and only w is constant almost everywhere.
The same result holds for the adjoint operator $S^* - I$.

Preliminaries and Examples

The previous corollary shows taking $w = \chi_{(0,1)}$ that, for $f \in L^2(\mathbb{R}^+)$, the restriction of either $Sf - f$, or $S^* f - f$, to the interval $(0,1)$ is not an isometry on $L^2(0,1)$, namely, there exists $f \in L^2(\mathbb{R}^+)$ such that

$$\int_0^1 \left(\int_t^\infty f(s) \frac{ds}{s} - f(t) \right)^2 dt \neq \int_0^1 f^2(t)\, dt,$$

(and similarly for $S - I$). However, it was proved by Brown et all that the restriction of the adjoint operator S^*, minus the identity, to functions $f \in L^2(0,1)$ is indeed an isometry:

$$\int_0^1 \left(\int_t^1 f(s) \frac{ds}{s} - f(t) \right)^2 dt = \int_0^1 f^2(t)\, dt.$$

Again, this result is false for $S - I$: there exists $f \in L^2(0,1)$ such that

$$\int_0^1 \left(\frac{1}{t} \int_0^t f(s)\, ds - f(t) \right)^2 dt \neq \int_0^1 f^2(t)\, dt,$$

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Preliminaries and Examples

Assume $T_K : L^2(X) \longrightarrow L^2(X)$ is bounded, and $Y \subset X$ is a given measurable subset. Using the orthogonal decomposition $L^2(X) = L^2(Y) \oplus L^2(Y^c)$, we define the operator

$$T_{K,Y} f(x) = \int_Y K(x,y) f(y)\, d\mu(y), \qquad x \in Y.$$

Theorem. Let T_K be an operator such that $T_K - I$ is an isometry in $L^2(X)$. Then, the following facts are equivalent:

(i) For every $f \in L^2(Y)$,

$$\|T_{K,Y} f - f\|_{L^2(Y)} = \|f\|_{L^2(Y)};$$

(ii) $Ker(P_{Y^c} \circ T_K \circ i_Y) = L^2(Y)$.

(iii) $K(x,y) = 0$, a.e. $(x,y) \in Y^c \times Y$.

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Preliminaries and Examples

Some other examples besides the Hardy operator and its adjoint:

- Convolution operators: $T_K f(x) = (K * f)(x)$, with $K \in L^1(\mathbb{R})$. We denote by $m = \hat{K}$, $T_K - I$ is an isometry in $L^2(X)$ if and only if

$$\|(T_K - I)f\|_{L^2(\mathbb{R})} = \|(m-1)\hat{f}\|_{L^2(\mathbb{R})} = \|\hat{f}\|_{L^2(\mathbb{R})}, \; f \in L^2(\mathbb{R}),$$

and, hence

$$|m(\xi) - 1| = 1, \quad \text{for every } \xi \in \mathbb{R}.$$

which is equivalent to

$$m(\xi)\bar{m}(\xi) = m(\xi) + \bar{m}(\xi), \quad \text{for every } \xi \in \mathbb{R},$$

or to

$$\int_{\mathbb{R}} K(z-x)K(z-y)\,dz = K(x-y) + K(y-x)$$

An example of kernel $K \in L^1(\mathbb{R})$ verifying this condition is

$$K(x) = \frac{\sin(2\pi^2 x)}{\pi x(2\pi x + 1)}.$$

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Preliminaries and Examples

- Separate variables kernels For integral operators whose kernel is a tensor product on each variable $K(x,y)=a(x)b(y)$, $(x,y)\in\mathbb{R}^+\times\mathbb{R}^+$, T_K-I is an isometry implies that

$$b(y)=\frac{2a(y)}{\|a\|^2_{L^2(\mathbb{R}^+)}}.$$

In this case, the same characterization is obtained if we replace the condition

$$\int_X K(z,x)K(z,y)\,d\mu(z)=K(x,y)+K(y,x).$$

on the kernel by the weaker diagonal condition:

$$\int_X K^2(z,x)\,d\mu(z)=2K(x,x),\quad \text{a.e. } x\in X.$$

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- $K(x,y) = \chi_E(x,y)$

Let us consider $K(x,y) = \chi_E(x,y)$, for some measurable set $E \subset \mathbb{R}^+ \times \mathbb{R}^+$, $T_K - I$ is a surjective isometry if and only if

$$|E^x \cap E^y| = |E_x \cap E_y| = \chi_E(x,y) + \chi_E(y,x), \quad \text{a.e. } (x,y) \in \mathbb{R}^+ \times \mathbb{R}^+,$$

where E_{x_0} and E^{y_0} denote the sections on each variable; that is,

$$E_{x_0} = \{y \in \mathbb{R}^+ : (x_0, y) \in E\}$$

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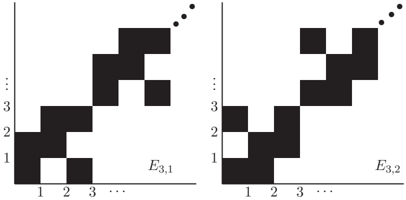
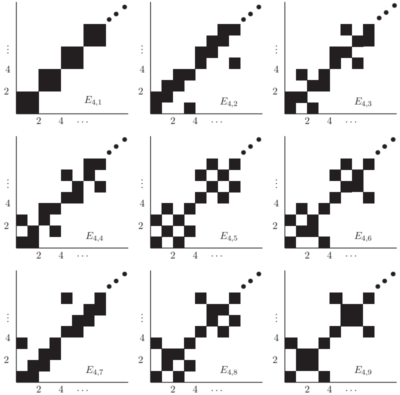
Preliminaries and Examples



Sets generated by a 3×3 period corresponding to a surjective isometry.

Preliminaries and Examples



Sets generated by a 4×4 period corresponding to a surjective isometry.

Preliminaries and Examples

- n-dimensional Hardy operator The Hardy operator has an n-dimensional extension S_n.
  For simplicity, let us assume $n = 2$, and for $s, t > 0$ define,

$$S_2 f(s,t) = \frac{1}{st} \int_0^s \int_0^t f(x,y)\, dy\, dx.$$

Contrary to the 1-dimensional case, it is easy to see that $S_2 - I$ is not an isometry on $L^2(\mathbb{R}^2_+)$. In fact, for $0 < s_1 < s_2$ and $0 < t_2 < t_1$, our condition reads,

$$\int_{\mathbb{R}^2_+} \frac{1}{xy} \chi_{(0,x)\times(0,y)}(s_1,t_1) \frac{1}{xy} \chi_{(0,x)\times(0,y)}(s_2,t_2)\, dy\, dx = \frac{1}{s_2 t_1},$$

but

$$\frac{1}{s_1 t_1} \chi_{(0,s_1)\times(0,t_1)}(s_2,t_2) + \frac{1}{s_2 t_2} \chi_{(0,s_2)\times(0,t_2)}(s_1,t_1) = 0.$$

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Main result

Definition Let (X, Σ, μ) be a measure space. We say that $\mathcal{E} \subset \Sigma$ is isoadmissible if:

(i) $\mu(E) < \infty$, for every $E \in \mathcal{E}$;

(ii) $\emptyset \in \mathcal{E}$;

(iii) $\mathcal{E}$ is stable under finite unions and intersections;

(iv) $\mathcal{U}_{\mathcal{E}} = \{\chi_E : \ E \in \mathcal{E}\}$ is a total set on $L^2(X, d\mu)$; that is,

$$\int_E f(x)\, d\mu(x) = 0, \text{ for all } E \in \mathcal{E} \Rightarrow f(x) = 0, \text{ a.e. } x \in X.$$

Main result

For $\mathcal{E} \subset \Sigma$ isoadmissible, we denote by $\mathcal{D}_{\mathcal{E}} = \{\chi_{E_1} + \chi_{E_2} : \ E_1, E_2 \in \mathcal{E}\}$ and

$$\mathcal{F}_{\mathcal{E}} = \Big\{ f \in L^2(X) : \big\{ x \in X : |f(x)| > t \big\} \in \mathcal{E}, \text{ for every } t > 0 \Big\}.$$

THEOREM. Let $\mathcal{E}$ be an isoadmissible set on (X, Σ, μ) and let T_K be an integral operator. Then, the following facts are equivalent:

(i) $T_K - I$ is an isometry on $L^2(X)$;

(ii) $T_K - I$ is an isometry restricted to $\mathcal{F}_{\mathcal{E}}$;

(iii) $T_K - I$ is an isometry restricted to $\mathcal{D}_{\mathcal{E}}$.

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(ii) $T_K - I$ is an isometry restricted to $\mathcal{F}_{\mathcal{E}}$;

(iii) $T_K - I$ is an isometry restricted to $\mathcal{D}_{\mathcal{E}}$.

Main result

Remark. It is relevant to observe that, in general, the diagonal condition

$$\int_X K^2(z,x)\,d\mu(z) = 2K(x,x), \quad \text{a.e. } x \in X,$$

is not enough to get an isometry for $T_K - I$.

Moreover, we need to evaluate $T_K - I$ on, at least, the sum of two characteristic functions of sets in $\mathcal{E}$ and this condition cannot be replaced by testing $T_K - I$ on just the characteristic functions of one set in $\mathcal{E}$.

Counterexample To see this, let us consider $X = \{1,2\}$ endowed with the counting measure on $\Sigma = \mathcal{P}(X)$, so that $L^2(X) = \mathbb{R}^2$. Choose $\mathcal{E} = \{\emptyset, \{1\}, \{1,2\}\}$, which is clearly an isoadmissible set.

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Example. $X = \mathbb{R}^+$ and $\mathcal{E} = \{(0,r) : r > 0\}$. It is clear that to show that $\mathcal{E}$ is isoadmissible because $\{\chi_E\}$ is a total set: if $\int_0^r f(x)\,dx = 0$, for every $r > 0$, then $f \equiv 0$, a.e. in $\mathbb{R}^+$. In this case $\mathcal{F}_{\mathcal{E}}$ is the cone of positive and decreasing functions in $\mathbb{R}^+$.

THEOREM. Let T_K be an operator given by a regulated function K. The following facts are equivalent:

(i) $T_K - I$ is an isometry on $L^2(\mathbb{R}^+)$;

(ii) $T_K - I$ is an isometry on the cone of decreasing functions $L^2_{\rm dec}(\mathbb{R}^+)$;

(iii) for almost all $(r,s) \in \mathbb{R}^+ \times \mathbb{R}^+$,

$$\int_0^\infty K(x,r)K(x,s)\,dx = K(r,s) + K(s,r).$$

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Remark

Note that, in general, the equivalence between (i) and (ii) in the previous theorem is not a direct consequence of the invariance by rearrangements property of the $L^2(\mathbb{R}^+)$-norm since, for an arbitrary integral operator T_K, bounded on $L^2(\mathbb{R}^+)$, the equality

$$\|T_K f - f\|_{L^2(\mathbb{R}^+)} = \|T_K f^* - f^*\|_{L^2(\mathbb{R}^+)}$$

fails in general for an $f \in L^2(\mathbb{R}^+)$.

Main result

Some other examples

$X = \mathbb{R}^n_+$ and

$$\mathcal{E} = \Big\{ D \subset \mathbb{R}^n_+ : \text{if } (x_1, \dots, x_n) \in D, \text{ then } (y_1, \dots, y_n) \in D, \\ \text{whenever } 0 < y_j \leq x_j,\ j = 1, \dots, n \Big\}.$$

Now, it is enough to observe that the condition of being a total set holds just considering sets that are intervals of the form:

$$D = (0, r_1) \times \cdots \times (0, r_n), \quad r_1, \dots, r_n > 0.$$

It is easy to see that $\mathcal{F}_{\mathcal{E}}$ is the cone of positive functions in $\mathbb{R}^n_+$, decreasing on each variable.

We observe that if we restrict to the family of cubes $\mathcal{E}_{\text{cubes}} = \{(0, r)^n : r > 0\}$, then $\mathcal{U}_{\mathcal{E}_{\text{cubes}}}$ is not total in $\mathbb{R}^n_+$.

This is an example that shows how all this theory can be extended to a more general context where an ordered structure can be defined: discrete setting like $\mathbb{N}$ or ordered trees ([B.,Soria](2008))

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