Bilinear Regular Operators on Quasi-Banach Lattices and Compactness

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joint work with Dicesar L. Fernandez
Multilinear operators arise naturally in many areas of classical and harmonic analysis, as well as functional analysis, including the theory of Banach operator ideals.
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Fundamental multilinear operators arising in harmonic analysis include convolutions, paraproducts and multilinear Fourier multiplier operators.
In the last years, several singular multilinear operators have been intensively studied and the research on bilinear Hilbert transform, originated by the work of M. Lacey and C. Thiele.
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has shown the need for the development of a systematic analysis of bilinear operators.
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- Introduction - Overview

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it is presented several important results about bilinear maps on products of normed vector lattices.
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Besides the classical works by Aoki, Rolewicz and Kalton, the studies of analytic and geometric aspects are one of the main issues for these spaces, with many results obtained recently.
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- Introduction

- In the current work, positive and regular bilinear operators on quasi-normed functional spaces are defined and some properties and characterizations on lattices and quasi-normed lattices are obtained.

- We introduce a variant definition of functional quasi-norm and prove some theorems characterizing compactness of bilinear operators.
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relations between positive and regular bilinear operators and their adjoint on normed functional spaces are proved.
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In what follows, the first definitions and results are presented. For a vector lattice $X$, the positive cone is denoted by $X_+$. 
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Definition 1. Let $X$, $Y$ and $Z$ be vector lattices. A bilinear operator $T : X \times Y \rightarrow Z$ is positive if given $x \in X_+$ and $y \in Y_+$, one has $T(x, y) \in Z_+$. 
Proposition 1. Let $X$, $Y$ and $Z$ be vector lattices and $T : X \times Y \to Z$ a bilinear positive operator. Then,

$$|T(x, y)| \leq T(|x|, |y|),$$

for all $(x, y) \in X \times Y$. 

- Positive Operators
Proposition 1. Let $X$, $Y$ and $Z$ be vector lattices and $T : X \times Y \rightarrow Z$ a bilinear positive operator. Then,

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for all $(x, y) \in X \times Y$.

Definition 2. Let $X$, $Y$ and $Z$ be ordered vector spaces. A bilinear operator $T : X \times Y \rightarrow Z$ is regular if it may be written as

$$T = T_1 - T_2,$$

where $T_1$ and $T_2$ are positive bilinear operators.
Theorem 1. Let $X$, $Y$ and $Z$ be vector lattices. A bilinear operator $T : X \times Y \to Z$ is regular if, and only if, there exists a positive bilinear operator $S : X \times Y \to Z$ such that

$$|T(x, y)| \leq S(|x|, |y|),$$

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Definition 3. An ordered set $X$ is Dedekind complete if every non-empty subset of $X$ that is bounded above admits a supremum (in $X$).
Theorem 2. Let $X$ and $Y$ be vector lattices and $Z$ a Dedekind complete vector lattice. A bilinear operator $T : X \times Y \to Z$ is regular if, and only if, for each $(u, v) \in X_+ \times Y_+$, there exists $\omega \in Z_+$ such that

$$T(x, y) \leq \omega,$$

for all $(x, y) \in X_+ \times Y_+$ with $0 \leq x \leq u$ and $0 \leq y \leq v$. 
Definition 4. A quasi-norm in a vector space $X$ is an application $\| \cdot \|$ of $X$ in $[0, \infty[$ such that, for $x, y \in X$ and $\lambda \in \mathbb{R}$, verifies the conditions...
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- **QN1)** $\|x\| = 0 \iff x = 0$;
- **QN2)** $\|\lambda x\| = |\lambda| \|x\|$;
- **QN3)** $\|x + y\| \leq C (\|x\| + \|y\|)$,

for some $C \geq 1$. 
A vector space $X$ endowed with a quasi-norm is called a **quasi-normed space**.
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A **quasi-Banach space** is a quasi-normed space which is complete in the topology generated by

\[ d(x, y) = ||x - y||. \]
- Quasi-Normed Lattices

A classic result is the Aoki-Rolewicz’s Theorem:
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**Theorem 3.** If $X$ is a quasi-normed space endowed with a quasi-norm $||.||$, there exists a constant $\alpha$, $0 < \alpha \leq 1$, and an equivalent quasi-norm $|||.||||$ such that

$$|||x + y|||^\alpha \leq |||x|||^\alpha + |||y|||^\alpha$$

for all $x, y$ in $X$. 
Definition 5. If a quasi-normed space is also a vector lattice \((X, \leq)\), we say \(X\) is a quasi-normed lattice if

\[ |x| \leq |y| \implies \|x\| \leq \|y\|. \]
**Definition 5.** If a quasi-normed space is also a vector lattice \((X, \leq)\), we say \(X\) is a **quasi-normed lattice** if

\[ |x| \leq |y| \implies \|x\| \leq \|y\|. \]

Besides, if a quasi-normed lattice is complete, we say it is a **quasi-Banach lattice**.
Theorem 4. Let $X$ and $Y$ be quasi-Banach lattices and $Z$ a quasi-normed lattice. If a bilinear operator $T : X \times Y \rightarrow Z$ is positive, then it is bounded.
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Corollary 1. In the conditions of the Theorem 4, if the operator $T : X \times Y \to Z$ is regular, it is also bounded.
We introduce now the function spaces which we will deal with. We define a variant general concept of functional quasi-norm which allow us to generalize several functional spaces.
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Let \((\Omega, \mu)\) a measure space. We denote by

\[ L^0_+ = L^0_+(\Omega, \mu) \]

the cone of real \(\mu\)-measurable, non negative and \(\mu\)-a.e. finite functions on \(\Omega\).
Definition 6. An application \( \rho : L^0_+ \rightarrow [0, \infty] \) is a functional quasi-norm if, for all \( f, g \in L^0_+ \), for all \( \lambda > 0 \) and for all subset \( D \subset \Omega \), with \( \mu(D) < \infty \), the following conditions are verified:
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- C1) \( \rho(f) = 0 \iff f = 0, \mu - \text{a.e.}; \)
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- **C1)** \( \rho(f) = 0 \iff f = 0, \mu - \text{a.e.}; \)
- **C2)** \( \rho(\lambda f) = \lambda \rho(f) \) for all \( \lambda > 0 \);
Definition 6. An application $\rho : L^0_+ \to [0, \infty]$ is a functional quasi-norm if, for all $f, g \in L^0_+$, for all $\lambda > 0$ and for all subset $D \subset \Omega$, with $\mu(D) < \infty$, the following conditions are verified:

- **C1)** $\rho(f) = 0 \iff f = 0$, $\mu$ – a.e.;

- **C2)** $\rho(\lambda f) = \lambda \rho(f)$ for all $\lambda > 0$;

- **C3)** $\rho(f + g) \leq C (\rho(f) + \rho(g))$, for some $C \geq 1$. 
\( 0 \leq g \leq f \quad \mu \text{ - a.e. } \implies \rho(g) \leq \rho(f) \)
- Operators and Functionals Spaces

- C4) $0 \leq g \leq f$ $\mu$ - a.e. $\implies \rho(g) \leq \rho(f)$;

- C5) $\rho(\chi_D) < \infty$;
- Operators and Functionals Spaces

- **C4)** \(0 \leq g \leq f\) \(\mu\) – a.e. \(\implies \rho(g) \leq \rho(f)\);

- **C5)** \(\rho(\chi_D) < \infty\);

- **C6)** \(\lambda \mu(\{x \in D \;;\; |f(x)| \geq \lambda\})^{1/p} \leq C' \rho(f)\), for some \(p > 0\) and constant \(C' > 0\), dependent of \(D\) and \(\rho\), and independent of \(f\).
The space $L^\infty = L^\infty(\Omega, \mu)$ is defined as the set of all measurable real functions on $\Omega$, which are essentially bounded, i.e. bounded up to a set of measure zero. For $f \in L^\infty$, its norm is given by:

$$||f|| = \inf\{a \in \mathbb{R} : \mu(\{t : f(t) > a\}) = 0\}.$$
We denote by $S = S(\Omega, \mu)$ the subclass of simple functions.
We denote by \( S = S(\Omega, \mu) \) the subclass of simple functions.

**Definition 7.** Let \( \rho \) be a functional quasi-norm in \( L^0_+(\Omega, \mu) \). The class of the functions \( f \in L^0 \) such that \( \rho(\|f\|) < \infty \) is denoted by \( X = X(\Omega, \mu, \rho) \).
Theorem 5. Let $\rho$ be a functional norm and $X = X(\Omega, \mu, \rho)$. For $f \in X$ let

$$\|f\|_X = \rho(|f|).$$

Then, $X$ is a quasi-normed vector subspace verifying the inclusions

$$S \subset X \hookrightarrow L^0.$$
Definition 8. Let $X = X(\Omega, \mu, \rho)$ a quasi-normed functional space. A function $f \in X$ has **absolutely continuous quasi-norm** if, given $\varepsilon > 0$ there exists $\delta > 0$ such that, $\mu(D) < \delta$ implies

$$\|f \chi_D\| < \varepsilon.$$
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We denote by $X_a$ the subspace of $X$ of all absolutely continuous quasi-normed functions.

$X$ has absolutely continuous quasi-norms if $X = X_a$. 
Definition 9. A family $\mathcal{M} \subset X$ has equi-absolutely continuous quasi-norm if, for all $\varepsilon > 0$ there exists $\delta > 0$ such that $\mu(D) < \delta$ implies

$$\|P_D f\| < \varepsilon,$$

for all $f \in \mathcal{M}$, where $P_D f(s) = f(s)$ if $s \in D$ and $P_D f(s) = 0$ if $s \notin D$. 
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Theorem 6. Let $X = X(\Omega, \mu, \rho_1)$ and $Y = Y(\Omega, \nu, \rho_2)$ be functional quasi-normed spaces. Each bounded bilinear operator $T$ acting from $X \times Y$ to $L^\infty$ is regular.
- Compactness Theorems

In what follows we give some characterizations of compact bilinear operators on the quasi-normed functional spaces.
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Let \( X = X(\Omega_1, \mu, \rho_1) \), \( Y = Y(\Omega_2, \nu, \rho_2) \) and \( Z = Z(\Omega_3, \nu, \rho_3) \) be quasi-normed functional spaces.
In what follows we give some characterizations of compact bilinear operators on the quasi-normed functional spaces.

Let $X = X(\Omega_1, \mu, \rho_1)$, $Y = Y(\Omega_2, \nu, \rho_2)$ and $Z = Z(\Omega_3, \upsilon, \rho_3)$ be quasi-normed functional spaces.

We denote by $\mathcal{B}il(X \times Y, Z)$ the family of all bounded bilinear operators from $X \times Y$ to $Z$. 
Definition 10. A bounded bilinear operator $T : X \times Y \rightarrow Z$ is compact in measure if the image \{ $T(u_n, v_n)$ \}, of any bounded sequence \{(u_n, v_n)\} of $X \times Y$, contains a Cauchy subsequence in respect to the measure $\nu$, that is,
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if \( \max\{\|u_n\|_X, \|v_n\|_Y\} \leq C \), then there exists a subsequence \( \{(u_{n_k}, v_{n_k})\} \) such that, given \( \varepsilon > 0 \) and \( \delta > 0 \), there exists \( N = N(\varepsilon, \delta) \) with
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$$\nu(\{s \in \Omega_3 : |T(u_{n_k}, v_{n_k})(s) - T(u_{m_k}, v_{m_k})(s)| > \varepsilon\}) < \delta$$

for all $n_k, m_k > N$. 
Theorem 7. Let $X$ and $Y$ quasi-normed functional spaces and suppose that $Z$ has absolutely continuous quasi-norms, i.e $Z = Z_\alpha$. Let $T : X \times Y \to Z$ be a bounded bilinear operator.
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$$\{ T(f, g) : \|f\|_X \leq 1, \|g\|_Y \leq 1 \}$$

have equi-absolutely continuous quasi-norms.
Theorem 8. Let $X$, $Y$ and $Z$ be quasi-normed functional spaces. Moreover, suppose that $Z$ has absolutely continuous quasi-norms, i.e $Z = Z_a$, and $\nu(\Omega_3) < \infty$. 
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A bilinear bounded operator $T : X \times Y \to Z$ is compact if, and only if, $T$ is compact in measure and satisfies

$$
\lim_{\nu(E) \to 0} \| P_E T \|_{Bil(X \times Y, Z)} = 0,
$$

where $E \subset Z$. 
Theorem 9. Let $X$, $Y$ and $Z$ be quasi-normed functional spaces, where $\mu(\Omega_1) < \infty$, $\nu(\Omega_2) < \infty$, $\upsilon(\Omega_3) < \infty$ and $Z$ has absolutely continuous quasi-norms, i.e $Z = Z_a$. 
Compactness Theorems

- **Theorem 9.** Let $X$, $Y$ and $Z$ be quasi-normed functional spaces, where $\mu(\Omega_1) < \infty$, $\nu(\Omega_2) < \infty$, $\nu(\Omega_3) < \infty$ and $Z$ has absolutely continuous quasi-norms, i.e. $Z = Z_a$. 

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A bilinear regular operator $T : X \times Y \to Z$ is compact if, and only if, $T$ is compact in measure and satisfies

$$\lim_{\nu(E) + \mu(D_1) + \nu(D_2) \to 0} \| P_E T(P_{D_1}, P_{D_2}) \|_{\mathcal{B}il(X \times Y, Z)} = 0,$$

where $D_1 \subset \Omega_1$, $D_2 \subset \Omega_2$ and $E \subset \Omega_3$. 
The present results are devoted to the relationships among the corresponding regular bilinear operators and their adjoints.
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Let us recall that Schauder’s well-known result states that an operator $T$ between Banach spaces is compact if, and only if, its adjoint, $T^*$, is compact.
- Compactness and adjoint operators

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- ideals of bilinear operators between Banach spaces, including the ideal of bilinear compact operators, i.e., $T \in \mathcal{B}ill(X \times Y, Z)$ such that $T(U_X \times U_Y)$ is relatively compact in $Z$. 
**Definition 11.** Given $T \in \mathcal{B}il(X \times Y, Z)$, the adjoint of $T$ is the linear map $T^\times : Z^* \rightarrow \mathcal{B}il(X \times Y)$ is given by

$$T^\times z^*(x, y) = z^*(T(x, y)), \quad (x, y) \in X \times Y.$$
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$\|T\| = \|T^\times\|.$
For $T^\times$ may be proved the analogue of Schauder’s theorem which states that if \( T \in \text{Bil}(X \times Y, Z) \), then \( T \) is compact if, and only if \( T^\times \) is compact.
For $T^\times$ may be proved the analogue of Schauder’s theorem which states that if $T \in \mathcal{B}il(X \times Y, Z)$, then $T$ is compact if, and only if $T^\times$ is compact.

And more, if $T \in \mathcal{B}il(X \times Y, Z)$ and $S \in L(Z, W)$, then

$$(ST)^\times = T^\times S^*$$

where $S^*$ is the classical linear adjoint.
- Compactness and adjoint operators

- From now, we are assuming that \( Z = L^p(\Omega_3, \nu) \) with \( 1 < p < \infty \).
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**Theorem 10.** Let $X$ and $Y$ be normed functional spaces, where $\mu(\Omega_1) < \infty$ and $\nu(\Omega_2) < \infty$, $\nu(\Omega_3) < \infty$. 
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Then, a bilinear bounded operator $T : X \times Y \to Z$ is compact if, and only if, $(T^\times)^* \text{ is compact in measure and satisfies }$
- Compactness and adjoint operators

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Then, a bilinear bounded operator $T : X \times Y \to Z$ is compact if, and only if, $(T^\times)^*$ is compact in measure and satisfies

$$\lim_{\mu(E) \to 0} \| T^\times P_E^* \| = 0,$$

where $E \subset \Omega_3$. 
In what follows, let $\Omega_1 = \Omega_2 = \Omega$ and $\mu = \nu$. 
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Definition 12. Given $D \subset \Omega$, we define

$$\overline{P}_D : \mathcal{B}il(X \times Y) \rightarrow \mathcal{B}il(X \times Y)$$
In what follows, let $\Omega_1 = \Omega_2 = \Omega$ and $\mu = \nu$.

**Definition 12.** Given $D \subset \Omega$, we define

$$
\overline{P}_D : \mathcal{B}il(X \times Y) \to \mathcal{B}il(X \times Y)
$$

such that, for $b \in \mathcal{B}il(X \times Y)$ then

$$
\overline{P}_D(b) \in \mathcal{B}il(X \times Y) \quad \text{and}
$$

$$
\overline{P}_D(b)(x, y) = b(P_Dx, P_Dy)
$$

for all $(x, y) \in X \times Y$. 
- Compactness and adjoint operators

- Proposition 2. $\overline{P}_D : \text{Bil}(X \times Y) \to \text{Bil}(X \times Y)$ is a bounded linear operator.
Compactness and adjoint operators

Proposition 2. \( \overline{P}_D : \mathcal{B}il(X \times Y) \rightarrow \mathcal{B}il(X \times Y) \) is a bounded linear operator.

Proposition 3. Considering the sequence of operators

\[ Z' \xrightarrow{P_E^*} Z' \xrightarrow{T^*} \mathcal{B}il(X \times Y) \xrightarrow{\overline{P}_D} \mathcal{B}il(X \times Y), \]
- Compactness and adjoint operators

- **Proposition 2.** \( \overline{P_D} : \text{Bil}(X \times Y) \rightarrow \text{Bil}(X \times Y) \) is a bounded linear operator.

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Z' \xrightarrow{P^*_E} Z' \xrightarrow{T^\times} \text{Bil}(X \times Y) \xrightarrow{\overline{P_D}} \text{Bil}(X \times Y),
\]

one has

\[
\|\overline{P_D} T^\times P^*_E\|_{L(Z', \text{Bil}(X \times Y))} = \|P_E T(P_D, P_D)\|_{\text{Bil}(X \times Y, Z)}.\]
Theorem 11. Let $X$ and $Y$ be normed functional spaces, where $\mu(\Omega) < \infty$ and $\nu(\Omega_3) < \infty$. 
- Compactness and adjoint operators

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- A bounded bilinear regular operator $T : X \times Y \rightarrow Z$ is compact if, and only if, $(T^\times)^*$ is compact in measure and satisfies
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$$\lim_{\nu(E) + \mu(D) \to 0} \| \overline{P} DT^\times P_E^* \|_{L(Z', \text{Bil}(X \times Y))} =$$

$$\lim_{\nu(E) + \mu(D) \to 0} \| P_E T(P_D, P_D) \|_{\text{Bil}(X \times Y, Z)} = 0.$$
Thank you for your attention!