Order-unit-metric spaces

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(joint work with Zafer Ercan)

Outline

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   - Basic properties
2. **Order-unit-metric spaces**
   - Order-units
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3. **Order-unit-topological spaces**
   - Topologies and convergence concepts
   - Metrizability
4. **Remarks on some applications**

It is well-known that on vector spaces, vector orderings and cones are in one-to-one correspondence.

This fact, alongside the equivalence of Archimedeanness and the closedness for the cone with non-empty interior of an ordered Hausdorff topological vector space, lies at the heart of great many results pertaining to the so-called cone metric spaces.

However, a typical adjustment of the concept of cone metric space, which we call order-unit-metric space, originates from and naturally fits into the framework of the rich theory of ordered vector spaces.

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Cone metric space

Let X be a non-empty set and E be an ordered real Banach space with $\operatorname{int}(K) \neq \varnothing$, where K is a closed cone in E. A function $d : X \times X \to E$ is called a ***cone metric*** if:

(i) $d(x, y) = 0$ if and only if $x = y$;

(ii) $d(x, y) = d(y, x)$ for all $x, y \in X$;

(iii) $d(x, y) \leqslant d(x, z) + d(y, z)$ for all $x, y, z \in X$.

In this case, the quadruple (X, E, K, d) is called a ***cone metric space***.

Let (X, E, K, d) be a cone metric space.

- The ordering "$\leqslant$" of the Banach space E in the above definition is generated by the cone K in the usual way.
- The family $\{B_{\ll}(x, r) : x \in X,\ 0 \ll r\}$, where $B_{\ll}(x, r) := \{y \in X : d(x, y) \ll r\}$, forms a basis for a topology on X, called the ***cone metric topology*** of the corresponding cone metric space. Here $x \ll y$ stands for the relation $y - x \in \operatorname{int}(K)$.
- The cone metric topology is metrizable (M. Khani & M. Pourmadhian–2011).

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Order-unit (Richard V. Kadison−1951)

Let E be an ordered real vector space. An element $0 < e \in E$ is called an ***order-unit*** if for each $x \in E$ there exists a $\lambda > 0$ such that $x \leqslant \lambda e$. The set of order units of E will be denoted by $\operatorname{ou}(E)$.

The inclusions $\operatorname{ou}(E) + \operatorname{ou}(E) \subseteq \operatorname{ou}(E)$ and $(0, \infty)\operatorname{ou}(E) \subseteq \operatorname{ou}(E)$ hold.

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- An ordered vector space E is called ***Archimedean*** if $x \leqslant 0$ whenever $\mathbb{N}x \leqslant y$ in E; further, E is called ***almost Archimedean*** if $x, y \in E$ and $-\varepsilon x \leqslant y \leqslant \varepsilon x$ for all $\varepsilon > 0$, then $x = 0$.
- The property of being almost Archimedean is not restrictive in the theory of ordered vector spaces: most of the work on cone metric spaces is being done for the so-called ***normal cone metric spaces***, which are Archimedean since the ordered Banach space in the definition admits an equivalent monotone norm; similarly, ordered Banach spaces with a closed cone are Archimedean as well.

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In what follows, the notation (E, K) will suggest that the vector ordering making the vector space E into an ordered vector space is generated by the cone K in E.

Theorem (G. Jameson–1970)

Let (E, K, τ) be an ordered topological vector space; that is, (E, K) is an ordered vector space and (E, τ) is a topological vector space. Then, the following are equivalent:

(i) $e \in \mathrm{int}(K)$.

(ii) $[-e, e]$ *is a neighborhood of zero.*

In this case, e is an order unit.

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Proof.

- If $[-e, e]$ is a neighborhood of zero, then since
$$e + [-e, e] = [0, 2e] \subseteq K,$$
we have $e \in \operatorname{int}(K)$.
- If $e \in \operatorname{int}(K)$, then there is a circled neighborhood of zero V with $e + V \subseteq K$. Then $V \subseteq [-e, e]$: indeed, for each $x \in V$ we have $-x \in V$, so $e + x$ and $e - x$ both lie in K. This shows that $x \in [-e, e]$, whence $[-e, e]$ is a neighborhood of zero. □

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Theorem (C.D. Aliprantis & R. Tourky–2007)

Let (E, K, τ) be an ordered Hausdorff topological vector space for which $\operatorname{int}(K) \neq \varnothing$. Then the cone K is Archimedean if and only if K is closed.

Theorem (C.D. Aliprantis & R. Tourky–2007)

Let (E, K, τ) be an Archimedean ordered Hausdorff topological vector space with $\operatorname{int}(K) \neq \varnothing$, and (E, τ) be completely metrizable. Then $\operatorname{int}(K) = \operatorname{ou}(K)$.

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In this case, the triple (X, E, d) is called an ***order-unit-metric space***.

- The map d takes values in the cone $E_+ := \{e \in E \mid e \geqslant 0\}$ of E; that is, $d(x,y) \geqslant 0$ for each $x, y \in X$.
- Every cone metric space is an order-unit-metric space, since $\operatorname{int}(E_+) \subseteq \operatorname{ou}(E)$.
- Recall that an ordered Banach space having a closed cone and order units is called a ***Krein space***, and for such a space E, the cone E_+ is Archimedean, and the order units are precisely the interior points of E_+.
- In particular, an ordered Banach space with a closed cone is a Krein space if and only if the interior of its cone is non-empty: ordered Banach spaces E appearing in the definition of cone metric spaces are, in fact, Krein spaces.

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Write $x \lll y$ if $y - x \in \mathrm{ou}\,(E)$, where (X, E, d) is an order-unit-metric space.

Theorem

The family $\{B_{\lll}(x, r) : x \in X,\ 0 \lll r\}$ for an order-unit-metric space (X, E, d), where $B_{\lll}(x, r) := \{y \in X : d(x, y) \lll r\}$, forms a basis for a topology on X.

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The cone metric topology can be recovered via the order-unit-topology.

Theorem

Let (X, E, d) be an order-unit-metric space and $e \in \mathrm{ou}\,(E)$ be fixed. Then the family $\{B_{\lll}(x, re) : 0 < r \in \mathbb{R}^+\}$ is a basis for the cone metric topology.

Proof. Let $u \in \mathrm{int}\,(E_+)$, which is an order-unit, be given. Then $re \leqslant u$ for some $r > 0$, from which it directly follows that $B_{\ll}(x, re) \subseteq B_{\lll}(x, u)$, whence the proof. □

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Let (X, E, K, d) be a cone metric space and (x_n) be a sequence in X.

- The sequence (x_n) is said to be $\ll$-u-***Cauchy***, where $0 \ll u$, if for every $\varepsilon > 0$ there exists a $k \in \mathbb{N}$ such that $d(x_n, x_m) \ll \varepsilon u$ for all $n, m \geqslant k$.
- The sequence (x_n) is called $\ll$-***Cauchy*** if it is $\ll$-u-Cauchy for all $u \in E$ with $0 \ll u$.
- The sequence (x_n) is said to be $\ll$-u-***convergent*** to $x \in X$, where $0 \ll u$, if for every $\varepsilon > 0$ there exists a $k \in \mathbb{N}$ such that $d(x_n, x) \ll \varepsilon u$ for all $n, m \geqslant k$; this is denoted by $x_n \to x \, (\ll u)$.
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For a fixed $e \in E$ with $0 \ll e$, the following facts are immediate:

$$\begin{aligned} &\text{(i) } (x_n) \text{ is } \ll\text{-Cauchy if and only if it is } \ll\text{-}e\text{-Cauchy.} \\ &\text{(ii) } x_n \to x\,(\ll) \text{ if and only if } x_n \to x\,(\ll e). \end{aligned} \quad (1)$$

Replacing the relation "$\ll$" (defined in cone metric spaces) in the above definitions by the relation "$\lll$", the corresponding concepts can be defined verbatim in order-unit-metric spaces.

Theorem

Let (X, E, d) be an order-unit-metric space, $e \in \mathrm{ou}\,(E)$, $x \in X$, and (x_n) be a sequence in X. Then, the following are equivalent:

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The principal machinery originates from the works of R. Kadison and F.F. Bonsall concerning representations of topological algebras and ideals in partially ordered spaces, respectively.

Let E be an Archimedean ordered real vector space with an order unit $e > 0$. We denote the (necessarily non-empty) set of positive functionals $f : E \to \mathbb{R}$ with $f(e) = 1$ by H_e.

As remarked in (Richard V. Kadison–1951), the space H_e with the weak*-topology is a compact Hausdorff space.

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Theorem (F.F. Bonsall−1954)

Let E be an Archimedean ordered vector space with an order unit $e > 0$. Then E is isomorphic to a subspace of $C(H_e)$ via the map

$$\pi : E \to C(H_e), \quad \pi(x)(f) := f(x).$$

If E is as in the previous Theorem, then, obviously, the map

$$p : E \to \mathbb{R}, \quad p(x) := \|\pi(x)\| \tag{2}$$

is a norm, where $\| \cdot \|$ is the usual supremum norm on $C(H_e)$.

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$$p : E \to \mathbb{R}, \quad p(x) := \|\pi(x)\| \tag{2}$$

is a norm, where $\|\cdot\|$ is the usual supremum norm on $C(H_e)$.

Theorem

Let (X, E, d) be an Archimedean order-unit-metric space and $e \in \operatorname{int}(K)$ be fixed. Then the order-unit-topology of (X, E, d) is metrizable.

Proof. Define the map $\overline{d} : X \times X \to \mathbb{R}$ by

$$\overline{d}(x, y) := p(d(x, y)), \tag{3}$$

where p is the norm given in (2). Clearly, p is a metric on X. We claim that the topology of the metric space $(X, \overline{d})$ and the order-unit-topology coincide.

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- One has

$$B_{\lll}(x,e)=\bigcup_{0<r<1}\{y:d(x,y)<re\}=\bigcup_{0<r<1}B_{\overline{d}}(x,r).$$

- For every $u\in E$ with $0\lll u$, the set $B_{\lll}(x,u)$ is $\overline{d}$-open.
- For each $0<r\in\mathbb{R}$, there exists $k\in\mathbb{N}$ such that

$$B_{\lll}(x,(1/k)\,e)\subseteq B_{\overline{d}}(x,r).$$

- Every $\overline{d}$-open ball is open in cone-metric-topology. □

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Call an order-unit-metric space (X, E, d) ***complete*** whenever the metric space $(X, \overline{d})$, with the metric $\overline{d}$ given in (3), is complete.

Theorem

Let (X, E, d) be an order-unit-metric space. Then, the following are equivalent:

(i) *(X, E, d) is complete.*

(ii) *If (x_n) is $\lll$-Cauchy then there exists $x \in X$ such that $(x_n) \to x(\lll)$.*

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With this in hand, the main result of (M. Khani & M. Pourmadhian−2011) can be recovered in a more general setting.

Corollary

Every cone metric space (X, E, K, d) for which E is completely metrizable is metrizable.

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Remarks on some applications

(I)

One of the areas where cone metric spaces is mostly used is the fixed point theory, where contraction mappings are the key ingredient.

Recall that a mapping $f : X \to X$, where (X, d) is a metric space, is called a ***contraction*** if there exists $0 < c < 1$ such that

$$d(f(x), f(y)) \leqslant c\, d(x, y)$$

for all $x, y \in X$.

If $f : X \to X$ is a contraction for a cone metric space (X, E, K, d), then f is also a contraction for the metric space $(X, \overline{d})$ with the metric $\overline{d}$ in (3).

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Relying on this observation, some basic facts that are true for metric spaces and are shown to be true for cone metric spaces with considerable effort can be directly obtained with the ultimate ease of order-unit-metric spaces.

The following is the main result of (Sh. Rezapour & R. Hamlbarani–2008):

Let (X, E, K, d) be a complete cone metric space and $T : X \to X$ be contractive with

$$d(T(x), T(y)) \leqslant k\, d(x, y)$$

for all $x, y \in X$, where $0 \leqslant k < 1$. Then, T has a unique fixed point in X, and for each $x \in X$, the iterative sequence $(T^n(x))$ converges to the fixed point.

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This is also the main result of (L.-G. Huang & X. Zhang−2007) for K normal, where a cone K is called ***normal*** if there exists $k > 0$ such that $0 \leqslant x \leqslant y$ implies $\|x\| \leqslant k\,\|y\|$. Note that in this case, using the metric $\overline{d}$, we have

$$\overline{d}(T(x), T(y)) \leqslant k\,\overline{d}(x, y)$$

for all $x, y \in X$. Thus, the above result is true for $E = \mathbb{R}$, which shows that the aforementioned results hold in a more general setting.

(II)

Observe that if

$$d(a,b) \leqslant \sum_{i=1}^{n} d(a_i,b_i),$$

where (X,E,K,d) is a cone metric space, then clearly

$$\overline{d}(a,b) \leqslant \sum_{i=1}^{n} \overline{d}(a_i,b_i)$$

holds. This implies that the results in (M. Abbas & G. Jungck−2008) concerning fixed points of contraction mappings can be shown to hold without any complexity, taking the ordered Banach space $E=\mathbb{R}$ with $K=[0,\infty)$; moreover, these results do not depend on the particular cone metric space.

(III)

It is shown in (A. Sönmez$-$2010) that cone metric spaces with a normal cone are paracompact: regarding the metrizability of cone metric spaces, this is a direct consequence of the classical result that every metric space is paracompact.

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