On some differential operators on hypercubes

Mirella Cappelletti Montano

Università degli Studi di Bari "A. Moro"

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Joint work with

F. Altomare
Dipartimento di Matematica
Università degli Studi di Bari "A. Moro"
Bari, Italy

Vita Leonessa
Dipartimento di Matematica, Informatica ed Economia
Università degli Studi della Basilicata
Potenza, Italy

Aim of the talk

Fix $N \geq 1$ and $l \in [0,2]$. We denote by $[0,1]^N$ the N-dimensional hypercube and we consider the elliptic second order differential operator

$$V_l : \mathscr{C}^2([0,1]^N) \longrightarrow \mathscr{C}([0,1]^N)$$

defined by setting

$$V_l(u)(x) := \frac{1}{2}\sum_{i=1}^{N} x_i(1-x_i)\frac{\partial^2 u}{\partial x_i^2}(x) + \sum_{i=1}^{N}\left(\frac{l}{2} - x_i\right)\frac{\partial u}{\partial x_i}(x)$$

$(u \in \mathscr{C}^2([0,1]^N),\ x = (x_i)_{1\leq i\leq N} \in [0,1]^N)$.

Aim of the talk

$(V_I, \mathscr{C}^2([0,1]^N))$ is closable and its closure is the generator of a Markov semigroup $(T_I(t))_{t\geq 0}$ on $\mathscr{C}([0,1]^N)$.

In some cases, this semigroup may be extended to a C_0-semigroup $(\widetilde{T}_I(t))_{t\geq 0}$ on $\mathscr{L}^p([0,1]^N)$, $p \geq 1$.

We also provide for a representation of $(T_I(t))_{t\geq 0}$ and $(\widetilde{T}_I(t))_{t\geq 0}$ (in the relevant norms) in terms of iterates of positive linear operators and we study some qualitative properties of those semigroups by means of the corresponding ones held by the approximating operators.

Our approach is not based on classical generation results, but on Approximation Theory.

Theorem (Schnabl)[1]

Let $(L_n)_{n\geq 1}$ be a sequence of linear contractions on a Banach space X and let $(\rho_n)_{n\geq 1}$ be a sequence of positive real numbers such that $\lim_{n\to\infty} \rho_n = 0$. Define the linear operator $A : D(A) \longrightarrow X$ by setting

$$A(f) := \lim_{n\to\infty} \rho_n^{-1}(L_n(f) - f)$$

for every $f \in D(A) := \left\{ g \in X \;\middle|\; \lim_{n\to\infty} \rho_n^{-1}(L_n(g) - g) \in X \right\}$. Moreover, assume that there exists a family $(F_i)_{i\in I}$ of finite-dimensional subspaces $D(A)$ such that $L_n(F_i) \subset F_i$ ($n \geq 1$, $i \in I$) and $\bigcup_{i\in I} F_i$ is dense in X.

[1] Über gleichmäßige Approximation durch positive lineare Operatoren, in Constructive Theory of Functions (Proc. Internat. Conf. Varna, 1970) 287-296; Izdat. Bolgar. Akad. Nauk, Sofia, 1972.

Theorem **(Schnabl)**

Then $(A, D(A))$ is the generator of a C_0-semigroup $(T(t))_{t\geq 0}$ on X such that, for every $t \geq 0$ and for every sequence $(k_n)_{n\geq 1}$ of positive integers such that $\lim_{n\to\infty} k_n\rho_n = t$, one gets

$$T(t)(f) = \lim_{n\to\infty} L_n^{k_n}(f)$$

for every $f \in X$.

Kantorovich operators in $[0,1]$ (1930)[2]

$$K_n : \mathcal{L}^1([0,1]) \to \mathscr{C}([0,1])$$

defined by setting, for every $f \in \mathcal{L}^1([0,1])$ *and* $x \in [0,1]$,

$$K_n(f)(x) := \sum_{k=0}^{n} (n+1) \left(\int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} f(t)\, dt \right) \binom{n}{k} x^k (1-x)^{n-k}.$$

$$\lim_{n\to\infty} K_n(f) = f \quad (f \in \mathscr{C}([0,1]))$$

$$\lim_{n\to\infty} K_n(f) = f \quad (f \in \mathscr{L}^p([0,1]),\ 1 \le p < +\infty).$$

[2] L.V. Kantorovich, *Sur certains développements suivant les polynômes de la forme de S. Bernstein I, II*, C.R. Acad. URSS (1930), 563-568 and 595-600.

A generalization by Altomare and Leonessa (2006)[3]

Let $(a_n)_{n\geq 1}$ and $(b_n)_{n\geq 1}$ be two sequences of real numbers such that $0 \leq a_n < b_n \leq 1$ $(n \geq 1)$; then, for every $n \geq 1$, define

$$C_n : \mathcal{L}^1([0,1]) \to \mathscr{C}([0,1])$$

by setting, for every $f \in \mathcal{L}^1([0,1])$ *and* $x \in [0,1]$,

$$C_n(f)(x) := \sum_{k=0}^{n} \left(\frac{n+1}{b_n - a_n} \int_{\frac{k+a_n}{n+1}}^{\frac{k+b_n}{n+1}} f(t)\,dt \right) \binom{n}{k} x^k (1-x)^{n-k}.$$

[3] F. Altomare and V. Leonessa, *On a sequence of positive linear operators associated with a continuous selection of Borel measures*, Mediterr. J. Math. **3** (2006), 363-382.

Kantorovich operators on $[0,1]^N$ (1993)[4]

Let $N \geq 2$.

$$K_n : \mathcal{L}^1([0,1]^N) \longrightarrow \mathscr{C}([0,1]^N)$$

defined by setting, for every $f \in \mathcal{L}^1([0,1]^N)$ *and* $x = (x_i)_{1 \leq i \leq N} \in [0,1]^N$,

$$K_n(f)(x) := \sum_{h_1,\dots,h_N=0}^{n} \prod_{i=1}^{N} \binom{n}{h_i} x_i^{h_i}(1-x_i)^{1-h_i}$$

$$\times (n+1)^N \int_{\frac{h_1}{n+1}}^{\frac{h_1+1}{n+1}} \cdots \int_{\frac{h_N}{n+1}}^{\frac{h_N+1}{n+1}} f(t_1,\dots,t_N)\, dt_1 \cdots dt_N.$$

[4] D.X. Zhou, *Converse theorems for multidimensional Kantorovich operators*, Anal. Math. **19** (1993), 85-100.

A generalization

Let $(a_n)_{n\geq 1}$ and $(b_n)_{n\geq 1}$ be two sequences of real numbers such that $0 \leq a_n < b_n \leq 1$ $(n \geq 1)$.

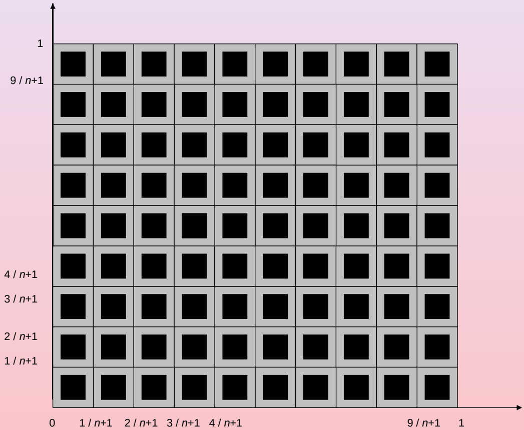
$$C_n : \mathcal{L}^1([0,1]^N) \longrightarrow \mathscr{C}([0,1]^N)$$

defined by setting, for every $f \in \mathcal{L}^1([0,1]^N)$ *and* $x = (x_i)_{1\leq i\leq N} \in [0,1]^N$,

$$C_n(f)(x) := \sum_{h_1,\dots,h_N=0}^{n} \prod_{i=1}^{N} \binom{n}{h_i} x_i^{h_i}(1-x_i)^{1-h_i}$$

$$\times \left(\frac{n+1}{b_n - a_n}\right)^N \int_{\frac{h_1+a_n}{n+1}}^{\frac{h_1+b_n}{n+1}} \cdots \int_{\frac{h_N+a_n}{n+1}}^{\frac{h_N+b_n}{n+1}} f(t_1,\dots,t_N)\, dt_1 \cdots dt_N.$$

Example for $n = 9$ and $N = 2$



Approximation properties in $\mathscr{C}([0,1]^N)$

Theorem

For every $f \in \mathscr{C}([0,1]^N)$

$$\lim_{n\to\infty} C_n(f) = f \quad \textit{uniformly on } [0,1]^N.$$

Proof

$\left\{\mathbf{1}, pr_1, \ldots, pr_N, \sum_{i=1}^{N} pr_i^2\right\}$ is a Korovkin set in $\mathscr{C}([0,1]^N)$; moreover, it is easy to check that, for every $n \geq 1$ and $i = 1, \ldots, N$,

$$C_n(\mathbf{1}) = \mathbf{1},$$

$$C_n(pr_i) = \frac{n}{n+1}\, pr_i + \frac{a_n + b_n}{2(n+1)}\, \mathbf{1}$$

and

$$C_n\left(\sum_{i=1}^{N} pr_i^2\right) = \frac{1}{(n+1)^2}\left(n^2 \sum_{i=1}^{N} pr_i^2 + n \sum_{i=1}^{N} pr_i(1-pr_i)\right.$$
$$\left. + n(a_n+b_n)\sum_{i=1}^{N} pr_i + N\,\frac{b_n^2 + a_n b_n + a_n^2}{3}\mathbf{1}\right).$$

Approximation properties in $\mathcal{L}^p([0,1]^N)$

Theorem

For every $n \geq 1$ and $p \in [1, +\infty[$, C_n is continuous from $\mathcal{L}^p([0,1]^N)$ into $\mathcal{L}^p([0,1]^N)$ and

$$\|C_n\|_{\mathcal{L}^p,\mathcal{L}^p} \leq \frac{1}{(b_n - a_n)^{N/p}}.$$

In particular, if

$$\sup_{n\geq 1} \frac{1}{(b_n - a_n)} < +\infty,$$

then, for every $f \in \mathcal{L}^p([0,1]^N)$,

$$\lim_{n\to\infty} C_n(f) = f \quad \mathit{in} \quad \mathcal{L}^p([0,1]^N).$$

Notation

Let K be a convex subset of $\mathbf{R}^N$, $f \in \mathscr{C}(K)$, $k \geq 1$ and $h \in \mathbf{R}^N$, $\|h\|_2 > 0$; we set

$$\Delta_h^k f(x) := \begin{cases} \displaystyle\sum_{l=0}^{k} (-1)^{k-l} \binom{k}{l} f(x+lh) & \text{if } x, x+kh \in K; \\ 0 & \text{otherwise.} \end{cases}$$

Fix $\delta > 0$. Then we set

$$\omega(f,\delta) := \sup\{|f(x) - f(y)| \ : \ x, y \in K, \|x-y\|_2 \leq \delta\}$$

first modulus of continuity of f with step δ.

Notation

Fix $k > 1$;

$$\omega_k(f,\delta) := \sup\{|\Delta_h^k f(x)| : x, x+kh \in K, \|h\|_2 \leq \delta\}$$

k-th modulus of smoothness of f with step δ.

If $f \in \mathcal{L}^p(K)$, $1 \leq p < \infty$, $k \geq 1$ and $\delta > 0$,

$$\omega_{k,p}(f,\delta) := \sup_{0<\|h\|_2\leq\delta} \left(\int_K |\Delta_h^k f(x)|^p \, dx \right)^{1/p}$$

k-th modulus of smoothness of f with step δ in $\mathcal{L}^p$.

Rate of convergence in $\mathscr{C}([0,1]^N)$[5]

Proposition

For every $f \in \mathscr{C}([0,1]^N)$ *and* $n \geq 1$,

$$\|C_n(f) - f\|_\infty \leq C\left(\frac{6N}{n+1}\|f\|_\infty + \omega_2\left(f, \sqrt{\frac{6N}{n+1}}\right)\right),$$

where the constant C *depends on* N*, only.*

[5] H. Berens and R. De Vore, *Quantitative Korovkin Theorems for positive linear operators on* L_p*-spaces*, Trans. Americ. Math. Soc. **245** (1978), 349-361.

Notation

Fix $p \in]1, \infty[$ and set $\alpha := 2 + N/p$ and $r := [\alpha] + 1$; we consider the **Lipschitz space**

$$Lip(\alpha, r; \mathcal{L}^p) := \left\{ f \in \mathcal{L}^p([0,1]^N) : \omega_{r,p}(f, \delta) = O(\delta^\alpha) \textit{ for every } \delta > 0 \right\}.$$

If $0 < \gamma \leq \alpha$, we set

$$\|f\|_p^\gamma := \|f\|_p + \sup_{0<t<1} t^{-\gamma} \omega_{r,p}(f, t).$$

Rate of convergence in $\mathcal{L}^p([0,1]^N)$ [6]

Theorem

For every $p \in [1, +\infty[$, *if* $f \in W_{2,\infty}([0,1]^N)$, *then, for every* $n \geq 1$,

$$\|C_n(f) - f\|_p \leq C \|f\|_{2,\infty} \frac{1}{n+1},$$

where the constant C *does not depend on* f.

Moreover, if $f \in \mathcal{L}^1([0,1]^N)$, *then, for every* $n \geq 1$,

$$\|C_n(f) - f\|_1 \leq C \left(\frac{6N}{n+1} \|f\|_1 + \omega_{N+2,1}\left(f, \left(\frac{6N}{n+1} \right)^{1/(N+2)} \right) \right).$$

[6] H. Berens and R. De Vore, *Quantitative Korovkin Theorems for positive linear operators on* L_p*-spaces*, Trans. Americ. Math. Soc. **245** (1978), 349-361.

Rate of convergence in $\mathcal{L}^p([0,1]^N)$

Finally, if $p \in]1, +\infty[$*, setting* $\alpha := 2 + \dfrac{N}{p}$*, if* $r = [\alpha] + 1$*,* $f \in Lip\,(\alpha, r; \mathcal{L}^p)$ *and* $0 < \gamma \leq \alpha$*, then, for every* $n \geq 1$*,*

$$\|C_n(f) - f\|_p \leq C\|f\|_p^{\gamma} \left(\frac{6N}{n+1}\right)^{\gamma/\alpha}.$$

Shape preserving properties

For every $m \geq 1$, let $\mathbb{P}_m$ be the space of all polynomials on $[0,1]^N$, having degree at most m.

Moreover, fix $M \geq 0$ and $0 < \alpha \leq 1$; then $Lip_M^1\alpha$ is the space of those $f \in \mathscr{C}([0,1]^N)$ such that, for every $x, y \in [0,1]^N$,

$$|f(x) - f(y)| \leq M\|x - y\|_1^{\alpha},$$

where $\|\cdot\|_1$ denotes the l_1-norm on $\mathbf{R}^N$, i.e., $\|x\|_1 := \sum_{i=1}^{N} |x_i|$ for every $x = (x_i)_{1 \leq i \leq N} \in \mathbf{R}^N$.

Shape preserving properties

For every $n, m \geq 1$,

$$C_n(\mathbb{P}_m) \subset \mathbb{P}_m$$

and, for every $n \geq 1$, $M \geq 0$ and $0 < \alpha \leq 1$,

$$C_n(Lip^1_M \alpha) \subset Lip^1_M \alpha.$$

Asymptotic formula

Assume that

$$\exists l := \lim_{n\to\infty}(a_n + b_n) \in \mathbf{R}.$$

Clearly, $0 \le l \le 2$.

Then for the sequence $(C_n)_{n\ge 1}$ and asymptotic formula, involving the elliptic second order differential operator $V_l : \mathscr{C}^2([0,1]^N) \longrightarrow \mathscr{C}([0,1]^N)$ defined, for every $u \in \mathscr{C}^2([0,1]^N)$ and $x = (x_i)_{1\le i\le N} \in [0,1]^N$, by

$$V_l(u)(x) := \frac{1}{2}\sum_{i=1}^{N} x_i(1-x_i)\frac{\partial^2 u}{\partial x_i^2}(x) + \sum_{i=1}^{N}\left(\frac{l}{2} - x_i\right)\frac{\partial u}{\partial x_i}(x),$$

holds true.

Asymptotic formula

Theorem

For every $u \in \mathscr{C}^2([0,1]^N)$,

$$\lim_{n\to\infty} n(C_n(u) - u) = V_I(u)$$

uniformly on $[0,1]^N$ *and, hence, in* $\mathcal{L}^p([0,1]^N)$.

Markov semigroup associated with the C_n's

Theorem

There exists a (unique) Markov semigroup $(T_I(t))_{t\geq 0}$ on $\mathscr{C}([0,1]^N)$ satisfying:

(1) *If $t \geq 0$ and $(k_n)_{n\geq 1}$ is a sequence of positive integers such that $\lim_{n\to\infty} k_n/n = t$, then $\lim_{n\to\infty} C_n^{k_n}(f) = T_I(t)(f)$ uniformly on $[0,1]^N$ for every $f \in \mathscr{C}([0,1]^N)$.*

(2) *Let $(A_I, D(A_I))$ be the generator od $(T_I(t))_{t\geq 0}$; then $\mathscr{C}^2([0,1]^N)$ is a core for $(A_I, D(A_I))$ and, hence, $(A_I, D(A_I))$ is the closure of $(V_I, \mathscr{C}^2([0,1]^N))$.*

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Markov semigroup associated with the C_n's

(3) $\mathbb{P} = \bigcup_{m \geq 0} \mathbb{P}_m$ *is a core for* $(A_I, D(A_I))$ *and* $T_I(t)(\mathbb{P}_m) \subset \mathbb{P}_m$ *for every* $t \geq 0$ *and* $m \geq 1$.

(4) $T_I(t)(Lip_M^1 \alpha) \subset Lip_M^1 \alpha$ *for every* $t \geq 0$*,* $M \geq 0$ *e* $0 < \alpha \leq 1$.

Remarks

The Abstract Cauchy Problem

$$\begin{cases} \dfrac{\partial u}{\partial t}(x,t) = A_I(u(\cdot,t))(x) & x \in [0,1]^N, \quad t \geq 0, \\ \\ u(x,0) = u_0(x) & u_0 \in D(A_I), \quad x \in [0,1]^N \end{cases}$$

admits a unique (classical) solution $u : [0,1]^N \times [0,+\infty[\to \mathbf{R}$, given by

$$u(x,t) = T_I(t)(u_0)(x)$$

for every $x \in [0,1]^N$ and $t \geq 0$.

In particular,

$$u(x,t) = T_I(t)(u_0)(x) = \lim_{n\to\infty} C_n^{[nt]}(u_0)(x),$$

uniformly w.r.t. $x \in [0,1]^N$.

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Remarks

Moreover,, since $A_l = V_l$ on $\mathbb{P}_m$, $m \geq 1$, if $u_0 \in \mathbb{P}_m$, then $u(x,t)$ is the (unique) classical solution to

$$\begin{cases} \dfrac{\partial u}{\partial t}(x,t) = \dfrac{1}{2}\sum_{i=1}^{N} x_i(1-x_i)\dfrac{\partial^2 u}{\partial^2 x_i}(x,t) + \sum_{i=1}^{N}\left(\dfrac{l}{2} - x_i\right)\dfrac{\partial u}{\partial x_i}(x,t) \\ \qquad\qquad x \in [0,1]^N, t \geq 0, \\ u(x,0) = u_0(x) \qquad x \in [0,1]^N \end{cases}$$

and $u(\cdot,t) \in \mathbb{P}_m$ for all $t \geq 0$.

Finally, if $u_0 \in D(A_l) \cap Lip_M^1\alpha$, then $u(\cdot,t) \in Lip_M^1\alpha$ for every $t \geq 0$.

Extending the semigroup in $\mathcal{L}^p([0,1]^N)$

Let V be the elliptic second order differential operator V_1, i.e.,

$$V(u)(x) = \frac{1}{2}\sum_{i=1}^{N} x_i(1-x_i)\frac{\partial^2 u}{\partial x_i^2}(x) + \sum_{i=1}^{N}\left(\frac{1}{2}-x_i\right)\frac{\partial u}{\partial x_i}(x)$$
$$= \sum_{i=1}^{N}\frac{\partial}{\partial x_i}\left(\frac{x_i(1-x_i)}{2}\frac{\partial u}{\partial x_i}\right)(x)$$

($u \in \mathscr{C}^2([0,1]^N)$ e $x = (x_i)_{1\le i\le N} \in [0,1]^N$).

We denote by $(T(t))_{t\ge 0}$ and $(A, D(A))$ the semigroup $(T_1(t))_{t\ge 0}$ and its relevant generator $(A_1, D(A_1))$.

Extending the semigroup in $\mathcal{L}^p([0,1]^N)$

Theorem

Let us assume either

(a) $a_n = 0$ *e* $b_n = 1$ *for every* $n \geq 1$

or

(b) the following properties hold

(i) $0 < b_n - a_n < 1$ *for every* $n \geq 1$;

(ii) *there exists* $\lim_{n\to\infty} a_n = 0$ *and* $\lim_{n\to\infty} b_n = 1$;

(iii) $M_1 := \sup_{n\geq 1} n(1 - (b_n - a_n)) < +\infty$.

Extending the semigroup in $\mathcal{L}^p([0,1]^N)$

Then, for every $p \geq 1$, $(T(t))_{t\geq 0}$ *may be extended to a (unique) positive* C_0*-semigroup* $(\widetilde{T}(t))_{t\geq 0}$ *on* $\mathcal{L}^p([0,1]^N)$ *such that*

$$\|\widetilde{T}(t)\|_{\mathcal{L}^p,\mathcal{L}^p} \leq e^{\omega_p t} \qquad t \geq 0,$$

where $\omega_p = 0$ *if assumption (a) holds true and* $\omega_p := NM_1M_2/p$*, if, alternatively, assumption (b) holds; in particular,*

$$M_2 := \sup_{n\geq 1} \frac{-\log(b_n - a_n)}{1-(b_n - a_n)} \geq 0.$$

Extending the semigroup in $\mathcal{L}^p([0,1]^N)$

Moreover, the generator $(\widetilde{A}, D(\widetilde{A}))$ *of* $(\widetilde{T}(t))_{t\geq 0}$ *is an extension (in* $\mathcal{L}^p([0,1]^N)$*) of* $(A, D(A))$ *and* $\mathscr{C}^2([0,1]^N)$ *is a core for* $(\widetilde{A}, D(\widetilde{A}))$ *and, hence,* $(\widetilde{A}, D(\widetilde{A}))$ *is the closure of* $(V, \mathscr{C}^2([0,1]^N))$ *in* $\mathcal{L}^p([0,1]^N)$*.*

Finally, if $f \in \mathcal{L}^p([0,1]^N)$*,* $t \geq 0$ *and* $(k_n)_{n\geq 1}$ *is a sequence of positive integers such that* $\lim_{n\to\infty} k_n/n = t$*, then, for every* $f \in \mathcal{L}^p([0,1]^N)$*,*

$$\lim_{n\to\infty} C_n^{k_n}(f) = \widetilde{T}(t)(f) \qquad \text{in} \quad \mathcal{L}^p([0,1]^N).$$

Example

Fix $\alpha \geq 1$ and for every $n \geq 1$ set

$$b_n := \frac{1}{2}\left(1 + \frac{1}{2n^\alpha} + \frac{n^\alpha}{n^\alpha + 1}\right)$$

and

$$a_n := \frac{1}{2}\left(1 + \frac{1}{2n^\alpha} - \frac{n^\alpha}{n^\alpha + 1}\right).$$

Then $0 \leq a_n < b_n \leq 1$ for any $n \geq 1$ and the sequences $(a_n)_{n\geq 1}$ and $(b_n)_{n\geq 1}$ satisfy assumption (b) in the previous result.

References

[1] F. Altomare, M. Cappelletti Montano and V. Leonessa, *On a generalization of Kantorovich operators on simplices and hypercubes*, Adv. Pure Appl. Math. **1** (3) (2010), 359-385.

[2] F. Altomare, M. Cappelletti Montano and V. Leonessa, *Iterates of multidimensional Kantorovich-type operators and their associated positive C_0-semigroups*, Stud. Univ. Babes-Bolyai Math. **56** (2) (2011), 219-236.