On the Lebesgue decomposition for non-additive functions[1]

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[1] joint work with Paolo de Lucia - Anna De Simone - Flavia Ventriglia, to appear.

The additive case on σ-algebras

Theorem (Lebesgue)

If $\mathcal{A}$ is a $\boldsymbol{\sigma}$**-algebra** *of sets and* $\boldsymbol{\mu} : \mathcal{A} \to \mathbb{R}$ *is a* $\boldsymbol{\sigma}$**-additive measure**,

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Remark: $\mu : \mathcal{A} \to \mathbb{R}$ σ**-additive** $\implies$ $\mathcal{N}(\mu)$ is a σ-ideal of $\mathcal{A}$

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Let $\mu : \mathcal{A} \to \mathcal{S}$*, where* $\mathcal{A}$ *is a* σ-**algebra** *and* $\mathcal{S}$ *a* **semigroup***, such that*

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Moreover, **if** φ **is additive**, **then** $\varphi = \varphi_{c'} + \varphi_c$.

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For $B \subseteq \mathcal{A}$ and $c \in \mathcal{A}$

$$B_c := \{x \in \mathcal{A} : x \wedge c \in B\} \quad \textit{trace on } c \textit{ of } B.$$

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Hence,

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(Also, B. Riečan - T. Neubrunn (1997)).

The additive case on orthomodular lattices

Let $L = (\mathrm{L}, 0, 1, \vee, \wedge, ')$ be an **OML**,

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2. μ is **null-additive** $\overset{def}{\Leftrightarrow}$ $\forall a \in L,\ b \in \mathcal{N}(\mu)$, s.t. $a \perp b$: $\mu(a \vee b) = \mu(a)$.

Clearly,

$$\mu \quad p\text{-function} \implies \mu \quad \text{null-additive}$$

► **Definitions 1-2 still works for non-additive** functions $\mu : L \to \mathcal{S}$, with $\mathcal{S} = (S, \mathcal{U})$ a **Hausdorff uniform space**, **on defining**

$$\mathcal{N}(\mu) = \{a \in L : \mu(x) = \mu(0) \text{ for every } x \in L \text{ such that } x \leq a\}.$$

Hereafter, $\mathcal{S} = (S, \mathcal{U})$ is a Hausdorff uniform space.

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When L is σ**-complete**, one also has

For $\mu : L \to \mathcal{S}$, where L is σ**-complete**.,

$$\mu \text{ is a } \sigma\text{–}p\text{-function} \overset{def}{\Longleftrightarrow} \begin{cases} \forall a \in L \text{ and orthogonal } (d_n)_{n\in\mathbb{N}} \text{ in } \mathcal{N}(\mu) \\ \mu(a) = \mu\left(a \vee (\vee_n d_n)\right) \end{cases}$$

In the same spirit of previous result on σ-complete OMLs

Lemma (C., de Lucia, De Simone, Ventriglia (2013))

Let $\mu : L \to \mathcal{S}$ *be* **null-additive**, *with* L σ*-complete.*

1. **If** $\mathcal{N}(\mu)$ **is a** σ*-complete p–ideal*, **then** μ **is a** σ-p**–function**.
2. **If** μ **is a** σ-p**–function**,**then** $\mathcal{N}(\mu)$ **is a** σ**-complete ideal**.
3. **If** μ **is a** σ-p**–function and weakly converse null–additive**, **then** $\mathcal{N}(\mu)$ **is a** σ**-complete** p**–ideal**.

s-outer functions

Let $\varphi : L \to \mathcal{S}$, with L an **OML** and $\mathcal{S} = (S, \mathcal{U})$ a Hausdorff uniform space.

φ is **s-outer** $\overset{def}{\Longleftrightarrow}$
$$\begin{cases} \forall U \in \mathcal{U} \ \ \exists V = V(U) \in \mathcal{U} \ \ s.t. \\ \ \ \forall a, b \in L, \ \ a \perp b : \\ \big(\varphi(a), \varphi(0)\big) \in V \implies \big(\varphi(a \vee b), \varphi(b)\big) \in U. \end{cases}$$

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Examples

Submeasures, **measuroids**, k-**triangular functions** and some **decomposible functions with respect to** t-**conorms** [H. Weber, Saeki, Guselńikov, Pap, S. Weber...] are s-**outer**

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Note

φ s-**outer** $\implies$ φ **null-additive**

Let $\varphi : L \to \mathcal{S}$ be **s-outer**. Then $\mathcal{D} = \{\varphi_1, \varphi_2\}$ is a **decomposition of φ** **IF and only IF**

1. $\varphi_i : L \to (S, \mathcal{U})$ is *s*-outer and $\varphi_i(0) = \varphi(0)$;
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Moreover, $\mathcal{D} = \{\varphi_{c'}, \varphi^{\perp}_{c'}\}$ is a decomposition of φ and such a decomposition is unique.

Idea of the proof.

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