Some geometrical properties of certain classes of uniform algebras

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POSTECH

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Schedule

1. Why a complex version of Urysohn Lemma ?
2. Numerical Index
3. Daugavet Property
4. Lushness
5. Approximate Hyperplane Series Property (AHSP)

Urysohn Lemma

Theorem (Urysohn Lemma)

A Hausdorff topological space X is normal if and only if given two disjoint closed subsets A and B of X there exists a continuous function $f : X \to [0,1]$ such that $f(x) = 0$ for all $x \in A$ and $f(y) = 1$ for all $y \in B$.

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Let K be a compact Hausdorff space.

Related to the numerical index, Daugavet property, Bishop-Phelps-Bollobás property, etc. several geometrical properties of the space $C(K)$ have been obtained by applying this classical Urysohn Lemma at one point or another.

Why a complex version of Urysohn Lemma

Most of those geometrical results on $C(K)$ could not be extended to a uniform algebra

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Form now on, $C(K)$ stands for the space of complex-valued continuous functions defined on a compact Hausdorff space K equipped with the supremum norm $\|\cdot\|_\infty$.

A uniform algebra is a closed subalgebra $A \subset C(K)$ that separates the points of K and contains the constant function $\mathbf{1}$

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A uniform algebra is a closed subalgebra $A \subset C(K)$ that separates the points of K and contains the constant function $\mathbf{1}$

For example consider the most representative case, the disk algebra $A(\mathbb{D})$ of functions continuous on the closed unit complex disk $\overline{\mathbb{D}}$ and holomorphic in the open disk $\mathbb{D}$ of $\mathbb{C}$.

Complex Version of Urysohn Lemma

Given $x \in K$, we denote by $\delta_x : A \to \mathbb{C}$ the evaluation functional at x given by $\delta_x(f) = f(x)$, for $f \in A$.

The natural injection $i : K \to A^*$ defined by $i(t) = \delta_t$ for $t \in K$ is a homeomorphism from K onto $(i(K), w^*)$.

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For a uniform algebra A of $C(K)$, if

$$S = \{x^* \in A^* : \|x^*\| = 1, x^*(\mathbf{1}) = 1\},$$

then the set $\Gamma_0(A)$ of all $t \in K$ such that δ_t is an extreme point of S is a boundary for A that is called the Choquet boundary of A.

Complex Version of Urysohn Lemma - Uniform Algebra

Theorem (Cascales, Guirao, and Kadets, 2012)

Let $A \subset C(K)$ be a uniform algebra for some compact Hausdorff space K and $\Gamma_0 = \Gamma_0(A)$.

Then, for every open set $U \subset K$ with $U \cap \Gamma_0 \neq \phi$ and $0 < \epsilon < 1$, there exist $f \in A$ and $t_0 \in U \cap \Gamma_0$ such that $f(t_0) = \|f\|_\infty = 1$, $|f(t)| < \epsilon$ for every $t \in K \backslash U$ and

$$f(K) \subset R_\epsilon = \{z \in \mathbb{C} : |\, Re(z) - 1/2| + (1/\sqrt{\epsilon})\, |Im(z)| \leq 1/2\}.$$

In particular,

$$|f(t)| + (1-\epsilon)|1 - f(t)| \leq 1, \text{ for all } t \in K.$$

Complex Version of Urysohn Lemma - Uniform Algebra

We apply this Urysohn type Lemma to extend results on $C(K)$ or $C(K, X)$ to A^X concerning the numerical index, Daugavet equation, lushness and the approximate hyperplane series property (in short $AHSP$).

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If A is a uniform algebra, then A^X is defined to be a subspace of $C(K, X)$ such that

$$A^X = \{f \in C(K, X) \ : \ x^* \circ f \in A \ \text{ for all } \ x^* \in X^*\}.$$

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Given $f \in A$ and $x \in X$, we define $f \otimes x \in C(K,X)$ by $(f \otimes x)(t) = f(t)x$ for $t \in K$. We write

$$A \otimes X = \{f \otimes x \ ; \ f \in A, \ x \in X\}.$$

From the definition of A^X we note that $A \otimes X \subset A^X$.

Main Results

We study some geometrical properties of certain classes of uniform algebras, in particular
the disk algebra $A_u(B_X)$ of all uniformly continuous functions on the closed unit ball and holomorphic on the open unit ball of a complex Banach space X.

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We study some geometrical properties of certain classes of uniform algebras, in particular
the disk algebra $A_u(B_X)$ of all uniformly continuous functions on the closed unit ball and holomorphic on the open unit ball of a complex Banach space X.

(1) The disk algebra $A_u(B_X)$ has k-numerical index 1 for every k, has both the lush property and the AHSP property.

(2) The algebra of the disk $A(\mathbb{D})$, and more in general any uniform algebra such that its Choquet boundary has no isolated points, has the polynomial Daugavet property.

Numerical Index

For a Banach space X, we write $\Pi(X)$ to denote the subset of $X \times X^*$ given by

$$\Pi(X) := \{(x, x^*) \ : \ x \in S_X, \ x^* \in S_{X^*}, \ x^*(x) = 1\}.$$

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Given a bounded function $\Phi : S_X \to X$, its *numerical range* is defined by

$$V(\Phi) := \{x^*(\Phi(x)) \ : \ (x, x^*) \in \Pi(X)\}$$

and its *numerical radius* is defined by

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Let us comment that for a bounded function $\Phi : \Omega \to X$, where $S_X \subset \Omega \subset X$, the above definitions are applied by just considering $V(\Phi) := V(\Phi|_{S_X})$.

Numerical Index

For $k \in \mathbb{N}$, we define

$$n^{(k)}(X) = \inf\{v(P) \ : \ P \in \mathcal{P}(^k X; X), \ \|P\| = 1\},$$

where $\mathcal{P}(^k X; X)$ is the space of all continuous k-homogeneous polynomials from X into X, and call it the *polynomial numerical index of order k of X*.

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If we consider elements of $\mathcal{A}_u(B_X, X)$ instead of continuous k-homogeneous polynomials, we can define the *analytic numerical index of X* by

$$n_a(X) = \inf\{v(f) \ : \ f \in \mathcal{A}_u(B_X, X), \ \|f\| = 1\}.$$

Since the space $\mathcal{P}(X;X)$ of all continuous polynomials from X into X is dense in $\mathcal{A}_u(B_X,X)$ we have that

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i.e. $n_a(X)$ can be called the "*non-homogeneous polynomial numerical index of X*".

Clearly,

$$n_a(X) \leq n^{(k)}(X)$$

for every $k \in \mathbb{N}$.

We also denote by $\mathcal{P}(X)$ the space of scalar-valued polynomials on X.

Numerical Index

Theorem

Suppose that A be a uniform algebra. Then $n^{(k)}(A^X) \geq n^{(k)}(X)$ *for every* $k \geq 1$ *and* $n_a(A^X) \geq n_a(X)$.

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Sketch of Proof ($n^{(k)}(X) \leq n^{(k)}(A^X)$ for every $k \geq 1$) Let $P \in S_{\mathcal{P}(^k A^X, A^X)}$ and $0 < \epsilon < 1$ be given. Choose $f_0 \in S_{A^X}$ so that $\|P(f_0)\| > 1 - \frac{\epsilon}{6}$. Find $t_1 \in \Gamma_0$ such that $\left\|P(f_0)(t_1)\right\| > 1 - \frac{\epsilon}{6}$. Since P is continuous at f_0, there exists $0 < \delta < 1$ such that $\|P(f_0) - P(g)\| < \frac{\epsilon}{6}$ for every $g \in A^X$ with $\|f_0 - g\| < \delta$.

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Let

$$W = \{t \in K \ : \ \|f_0(t) - f_0(t_1)\| < \delta/6, \ \|P(f_0)(t) - P(f_0)(t_1)\| < \epsilon/3\}.$$

This set is open in K and $t_1 \in W \cap \Gamma_0$.

Sketch of Proof

By the complex version of Urysohn Lemma
there exist a function $\phi\ :\ K \to \overline{\mathbb{D}}$ and $t_0 \in W \cap \Gamma_0$
such that $\phi \in A$, $\phi(t_0) = 1$, $|\phi(w)| < \frac{\delta}{6}$ for every $w \in K \backslash W$, and

$$|\phi(t)| + \big(1 - \frac{\delta}{6}\big)|1 - \phi(t)| \leq 1$$

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Define $\Psi(x) = \big(1 - \frac{\delta}{6}\big)(1 - \phi)f_0 + \phi x \in A^X$ for $x \in X$.

Sketch of Proof

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there exist a function $\phi \;:\; K \to \overline{\mathbb{D}}$ and $t_0 \in W \cap \Gamma_0$
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Define $Q \in \mathcal{P}(^k X; X)$ by

$$Q(x) = P\big(\Psi(x)\big)(t_0), \;\; (x \in X). \tag{1}$$

THEOREM

Let A be a uniform algebra and X be a Banach space. Assume that A^X has the following property: For every $P \in \mathcal{P}(^k X; X)$ and $t \in K$, $Q : A^X \longrightarrow C(K, X)$ where $Q(f)(t) = P(f(t))$ satisfies that $Q(f) \in A^X$ for every $f \in A^X$. Then $n^{(k)}(A^X) = n^{(k)}(X)$.

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Corollary

For any Banach space X, and $k \geq 1$, the following hold.

1. *For a uniform algebra A we have $n^{(k)}(A) = 1$ for every $k \geq 1$ and $n_a(A) = 1$.*
2. *$n^{(k)}(\mathcal{A}(\mathbb{D}^n, X)) = n^{(k)}(X)$ and $n_a(\mathcal{A}(\mathbb{D}^n, X)) = n_a(X)$ for every $n \in \mathbb{N}$.*
3. *$n^{(k)}(\mathcal{A}_u(B_X)) = 1$ for every $k \geq 1$ and $n_a(\mathcal{A}_u(B_X)) = 1$.*

A Banach space X is said to have the Daugavet property if the norm identity, so called the Daugavet equation,

$$\|Id + T\| = 1 + \|T\|$$

holds for every rank-one operator (and hence for every weakly compact operator) $T \in L(X)$.

Polynomial Daugavet Property

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Theorem

Suppose that A is a uniform algebra whose Choquet boundary has no isolated points. For every $P \in S_{\mathcal{P}(A^X)}$, $f_0 \in S_{A^X}$ and $\epsilon > 0$, there exist some $\omega \in S_{\mathbb{C}}$ and $g \in B_{A^X}$ such that $Re\omega P(g) > 1 - \epsilon$ and $\|f_0 + \omega g\| > 2 - \epsilon$.

Polynomial Daugavet Property

Corollary

If A is a uniform algebra whose Choquet boundary has no isolated points, then every weakly compact $P \in \mathcal{P}(A^X; A^X)$ satisfies the Daugavet equation which means that A^X has the polynomial Daugavet property.

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Since given a ball U in $\mathbb{C}^n$ or the polydisk $\mathbb{D}^n$ the Choquet boundary of $\mathcal{A}(U)$ does not have isolated points,

Corollary

If U is a ball in $\mathbb{C}^n$ or the polydisk $\mathbb{D}^n$, then $\mathcal{A}(U, X)$ has the polynomial Daugavet property for every complex Banach space X. In particular, $\mathcal{A}(U)$ has the polynomial Daugavet property.

Lushness

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The lushness has been known to be the weakest among quite a few isometric properties in the literature which are sufficient conditions for a Banach space to have the numerical index 1.

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The lushness has been known to be the weakest among quite a few isometric properties in the literature which are sufficient conditions for a Banach space to have the numerical index 1.

A Banach space X is said to be *lush* if for every $x, y \in S_X$ and for every $\epsilon > 0$ there is a slice

$$S = S(B_X, x^*, \epsilon) = \{x \in B_X \ : \ Rex^*(x) > 1 - \epsilon\}, \ x^* \in S_{X^*}$$

such that $x \in S$ and $dist(y, aconv(S)) < \epsilon$.

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Corollary

If X is lush, then $\mathcal{A}_u(B_Y)$ is lush for any Banach space Y.

AHSP

A Banach space X is said to have the *AHSP* if for every $\epsilon > 0$ there exist $\gamma(\epsilon) > 0$ and $\eta(\epsilon) > 0$ with $\lim_{\epsilon\to 0^+} \gamma(\epsilon) = 0$ such that for every sequence $(x_k)_{k=1}^{\infty} \subset B_X$ and for every convex series $\sum_{k=1}^{\infty} \alpha_k$ satisfying

$$\|\sum_{k=1}^{\infty} \alpha_k x_k\| > 1 - \eta(\epsilon)$$

there exist a subset $A \subset \mathbb{N}$, a subset $\{z_k : k \in A\} \subset S_X$ and $x^* \in S_{X^*}$ such that
(i) $\sum_{k\in A} \alpha_k > 1 - \gamma(\epsilon)$
(ii) $\|z_k - x_k\| < \epsilon$ for all $k \in A$, and
(iii) $x^*(z_k) = 1$ for all $k \in A$.

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(ii) $\|z_k - x_k\| < \epsilon$ for all $k \in A$, and
(iii) $x^*(z_k) = 1$ for all $k \in A$.

This property was introduced to characterize a Banach space X such that the pair (ℓ_1, X) has the Bishop-Phelps-Bollobás property for operators.

AHSP

Since every lush space has the *AHSP*, We have the following.

Corollary

Suppose that A is a uniform algebra.

1. *If X is lush, then A^X has the AHSP. In particular, A has the AHSP.*
2. *If A^X is lush, then X has the AHSP.*

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Corollary

$\mathcal{A}_u(B_Y)$ has the AHSP for any Banach space Y.

Thank You

Thank you for your attention.