The Dedekind Completion of $C(X)$

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The Dedekind completion of a Riesz space

A Riesz space K is called **Dedekind complete** if any **order bounded from above** subset of K has a **supremum** (equivalently, any order bounded from below subset of K has an infimum).

Definition

Let L be a Riesz space.
A **Dedekind complete** Riesz space L^δ is called a **Dedekind completion** of the Riesz space L if:

(i) There exists an **one-to-one Riesz homomorphism** of L into L^δ, $\phi : L \longrightarrow L^\delta$.

(ii) If we identify the Riesz subspace $\phi(L)$ of $L^\delta \subset L$, then, for every element $\widehat{f} \in L^\delta$, we have

$$\bigvee\{g : g \in L,\ g \leq \widehat{f}\} = \widehat{f} = \bigwedge\{g : g \in L,\ g \geq \widehat{f}\}. \tag{1}$$

Condition (ii) says two facts:

(a) $\widehat{f} = \bigvee\{g : g \in L,\, g \leq \widehat{f}\}$ shows that L is **order dense** in L^δ.

(b) $\widehat{f} = \bigwedge\{g : g \in L,\, g \geq \widehat{f}\}$ shows that the **ideal generated** by L in L^δ is all L^δ. Symbolically, $I(L) = L^\delta$.

It is well known that:

Theorem

Every Archimedean Riesz space L has a Dedekind completion L^δ.

W. A. J. Luxemburg and A. C. Zaanen, *Riesz Spaces*, vol. I, North-Holland Math. Library, North-Holland, Amsterdam-London, 1971, pp. 191-194.

Let $L = C(X)$, the Archimedean Riesz space of all real-valued continuous functions on a topological space X.
$C(X)^{\delta} = ?$

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- 2010 - R. Becker - **upper** semicontinuous functions

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2. To give proofs using only Riesz space techniques.
3. To show the relations between this construction of $C(X)^{\delta}$ and the other constructions existing in literature.

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- **Baire's operators:** $I, S : \mathcal{B}(X) \longrightarrow \mathcal{B}(X)$, where

$$I(f)(x) := \sup_{V\in\mathcal{V}_x} \inf_{y\in V} f(y), \quad \text{the } \textbf{lower limit function},$$

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- **Kaplan's operators:** $\ell, u, \delta : \mathcal{B}(X) \longrightarrow \mathcal{B}(X)$

$$\ell(f) := \bigvee\{g \in C(X) : g \leq f\}, \quad u(f) := \bigwedge\{g \in C(X) : g \geq f\},$$

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- X **completely regular** $\Rightarrow I(f) = \ell(f)$, $S(f) = u(f)$, $\forall f \in \mathcal{B}(X)$.

Some properties of Kaplan's operators ℓ and u

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- In consequence,

$$\ell(f) - u(g) \le \ell(f-g) \le \begin{matrix} u(f) - u(g) \\ \ell(f) - \ell(g) \end{matrix} \le u(f-g) \le u(f) - \ell(g).$$

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- The operators ℓ and u have the following lattice properties:

$$\ell(f \wedge g) = \ell(f) \wedge \ell(g), \quad u(f \vee g) = u(f) \vee u(g)$$

$$f \wedge g = 0 \Rightarrow \ell(f) \wedge u(g) = 0$$

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- For $f, g \in \mathcal{B}(X)$ we have:

$$\delta(f) - \delta(g) \leq \begin{matrix} \delta(f+g) \\ \delta(f-g) \end{matrix} \leq \delta(f) + \delta(g)$$

$$|\delta(f) - \delta(g)| \leq \begin{matrix} \delta(f+g) \\ \delta(f-g) \end{matrix}$$

$$\begin{matrix} \delta(f \vee g) \\ \delta(f \wedge g) \end{matrix} \leq \delta(f) \vee \delta(g)$$

$$\delta(f+g) \leq \delta(f \vee g) + \delta(f \wedge g) \leq \delta(f) + \delta(g) \tag{2}$$

$$\delta(f) = \delta(f^+) + \delta(f^-)$$

$$\delta(f) \leq 2u(|f|)$$

Kaplan's operators and uniform norm

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For $f, g \in \mathcal{B}(X)$, we have:

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- The sets $\mathcal{L}_{sc}$ and $\mathcal{U}_{sc}$ are **Dedekind complete lattices** in which the supremum and the infimum of any nonempty order bounded subset $\{f_\gamma\}_{\gamma\in\Gamma}$ are given by the formulae:

$$\bigvee_{\mathcal{L}} f_\gamma = \bigvee f_\gamma, \qquad \bigwedge_{\mathcal{L}} f_\gamma = \ell(\bigwedge f_\gamma),$$

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- If the set $\{f_\gamma\}_{\gamma\in\Gamma}$ is **finite**, then $\bigwedge_{\mathcal{L}} f_\gamma = \bigwedge f_\gamma$ and $\bigvee_{\mathcal{U}} f_\gamma = \bigvee f_\gamma$.

Lattices of normal semicontinuous functions

- $\mathcal{N}\mathcal{L}_{sc}(X) = \{f \in \mathcal{L}_{sc} \mid \ell u(f) = f\}$ - normal lower semicontinuous functions

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Pointwise discontinuous functions

- For $f \in \mathcal{B}(X)$, C_f denotes set of **points of continuity** of f. C_f is a G_δ set:

$$C_f = \{x \in X : \delta(f)(x) = 0\} = \bigcap_{n=1}^{\infty} \{x \in X : \delta(f)(x) < 1/n\}.$$

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- A function $f \in \mathcal{B}(X)$ is called **pointwise discontinuous** on X if it is continuous on a **dense** subset of X. $C_d(X)$ denotes the set of all these functions.

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- $\pi : C_d(X) \longrightarrow C_d(X)/\sim$ is the quotient map: $\pi(f) = \widehat{f}$.

The result $C(X)^\delta = C_d(X)/\sim$.

Theorem

Let X be a **compact** *space. Then:*
(i) The **Dedekind completion** *of the Riesz space $C(X)$ is*
$C(X)^\delta := C_d(X)/\sim$.
(ii) Endowed with the quotient of the uniform norm, $C(X)^\delta$ is a **Banach lattice**.

$$C(X) \hookrightarrow C_d(X) \xrightarrow{\pi} C(X)^\delta = C_d(X)/\sim .$$

Obvious $\widehat{g} = \pi(g) = \{g\}$, so $C(X)$ can be identified with a Riesz subspace $C(X)^\delta$. It remains to prove:

(A) $C(X)^\delta$ is a Dedekind complete Riesz space.

(B) For every $\widehat{f} \in C(X)^\delta$ we have

$$\bigvee\{g \in C(X) : g \leq \widehat{f}\} = \widehat{f} = \bigwedge\{g \in C(X) : g \geq \widehat{f}\}.$$

(C) $C(X)^\delta$ is a Banach lattice.

The characterization of a pointwise discontinuous function with Kaplan's operators

$$f \in C(X) \Longleftrightarrow \delta(f) = 0$$

$$f \in C_d(X) \Longleftrightarrow \ell\left(\delta\left(f\right)\right) = 0.$$

Theorem

Let X be a compact space and $f \in \mathcal{B}(X)$. The following statements are equivalent.
(i) f is **pointwise discontinuous** *on X, that is, $\overline{C_f} = X$.*
(ii) For every real number $\lambda > 0$ the set $A_\lambda(f) = \{x \in X : \delta(f)(x) \geq \lambda\}$ is nowhere dense.
(iii) f is continuous on a comeager set $X \setminus C_f = \cup_{n=1}^{\infty} A_{1/n}(f)$.
(iv) $\ell\left(\delta\left(f\right)\right) = 0$.

Theorem

$C_d(X)$ is a **norm closed Riesz subspace** *of $\mathcal{B}(X)$, hence a* **Banach lattice**.

Proof. If $f, g \in C_d(X)$, then $\ell(\delta(f)) = 0$, $\ell(\delta(g)) = 0$.

$$\begin{aligned} 0 &\leq \ell[\delta(f+g)] \leq \ell[\delta(f \vee g) + \delta(f \wedge g)] \leq \ell\ell[\delta(f) + \delta(g)] \leq \\ &\leq \ell[\underbrace{\ell(\delta(f))}_{=0} + u(\delta(g))] = \ell(\delta(g)) = 0. \end{aligned}$$

Hence $\ell[\delta(f+g)] = 0$, $\ell[\delta(f \vee g)] = 0$, $\ell[\delta(f \wedge g)] = 0 \Rightarrow$

$$f + g, f \vee g, f \wedge g \in C_d(X).$$

$$\ell(\delta(\lambda f)) = \ell(|\lambda| \delta(f)) = |\lambda| \ell(\delta(f)) = 0 \Rightarrow \lambda f \in C_d(X).$$

If $(f_n) \subset C_d(X)$ such that $\|f_n - f\| \rightharpoonup 0$ for some $f \in \mathcal{B}(X)$, then

$$\| \underbrace{\ell(\delta(f_n))}_{=0} - \ell(\delta(f)) \| \leq \|\delta(f_n) - \delta(f)\| \leq 2\|f_n - f\| \to 0 \Rightarrow f \in C_d(X).$$

The ideal of rare functions

A function $f \in \mathcal{B}(X)$ is called **rare** if $\ell u\left(|f|\right) = 0$.
$\mathcal{R}a(X)$ or $\mathcal{R}a$ denotes the set of all rare functions.

Theorem

$f \in \mathcal{R}a(X) \Leftrightarrow \ell\left(\delta(f)\right) = 0, \quad \ell\left(|f|\right) = 0.$

$$f \in \mathcal{R}a(X) \Leftrightarrow f(x) = 0, \quad \text{for all } x \in C_f$$
$$\mathcal{R}a(X) \subset C_d(X)$$

Theorem

$\mathcal{R}a(X)$ *is a* **norm closed Riesz ideal** *of* $C_d(X)$.

$$f \sim g \Leftrightarrow f - g \in \mathcal{R}a$$

Theorem

$C(X)^\delta = C_d(X)/\mathcal{R}a$ *is a* **Banach lattice**.

Theorem

For $f \in C_d(X)$ *a* **pointwise discontinuous** *function* f, *the following are equivalent:*

(i) $\ell\left(\delta\left(f\right)\right) = 0.$

(ii) $\ell u\left(\delta\left(f\right)\right) = 0.$

(iii) $\ell u\left[u\left(f\right) - f\right] = 0 \Leftrightarrow f \sim u(f).$

(iv) $\ell u\left[f - \ell\left(f\right)\right] = 0 \Leftrightarrow f \sim \ell(f).$

(v) $\ell u(f) = \ell u \ell(f)$, *that is* f *is Nakano quasicontinuous*

(vi) $u\ell(f) = u\ell u(f).$

(vii) $\ell u\left[u(f) - u\ell\left(f\right)\right] = 0 \Leftrightarrow u(f) \sim u\ell\left(f\right)$

(vii) $\ell u\left[\ell u(f) - \ell\left(f\right)\right] = 0 \Leftrightarrow \ell\left(f\right) \sim \ell u(f).$

Hence, for $f \in C_d(X) \Rightarrow f \sim \ell(f) \sim u(f) \sim \ell u(f) \sim u\ell\left(f\right).$
In other words, $\widehat{f}$ contains lsc, usc, nlsc and nusc functions.

Regular pair

Definition

A pair of functions $(\underline{f}, \overline{f})$ is called **regular** if $\underline{f} \in \mathcal{L}_{sc}$, $\overline{f} \in \mathcal{U}_{sc}$ and

$$\underline{f} \leq \overline{f}, \qquad u(\underline{f}) = \overline{f}, \qquad \ell(\overline{f}) = \underline{f}.$$

If $(\underline{f}, \overline{f})$ is a regular pair, then:
(i) The lower function $\underline{f} \in \mathcal{N}\mathcal{L}_{sc}$ and the upper function $\overline{f} \in \mathcal{N}\mathcal{U}_{sc}$.
(ii) $\ell(\overline{f} - \underline{f}) = 0$, that is, $\overline{f} \sim \underline{f}$.
(iii) The interval-valued function $\overline{\underline{f}} : X \longrightarrow \mathbb{IR}$, $x \longrightarrow [\underline{f}(x), \overline{f}(x)]$ is Hausdorff continuous (in the sense of Sendov).

Theorem

A regular pair is a Dedekind cut in $C(X)$.

N. Dăneț, *Dedekind cuts in C(X)*, Marcinkiewicz Centenary Volume, Polish Academy of Sciences, Banach Center Publications, Vol. 95, Warszawa, 2011, 287-297.

$$f \in C_d(X) \Rightarrow f \sim \ell(f) \sim u(f) \sim \ell u(f) \sim u\ell(f)$$

Theorem

(i) Every equivalence class $\widehat{f} \in C(X)^\delta = C_d(X)/\mathcal{R}a$ *contains* **exactly one regular pair** $(\underline{f}, \overline{f})$, *namely*

$$\underline{f} = \ell u(f), \quad \overline{f} = u\ell(f).$$

(ii) $\underline{f}$ *is the* **largest** *lower semicontinuous function in* $\widehat{f}$, *and* $\overline{f}$ *is the* **smallest** *upper semicontinuous function in* $\widehat{f}$.

S. Kaplan, *The bidual of C(X)* I, North-Holland Mathematics Studies 101, Amsterdam, 1985, p. 383.

The equivalence relation $f \sim g \Leftrightarrow f - g \in \mathcal{R}a$ can be define on $\mathcal{B}(X)$.

Theorem

The quotient map $\pi : \mathcal{B}(X) \longrightarrow \mathcal{B}(X)/\mathcal{R}a$ *has the following* **partial order continuity**.

(i) If $g = \bigwedge_{\gamma} g_{\gamma}$, *where* $\{g_{\gamma}\}_{\gamma \in \Gamma}$ *is a subset of* $\mathcal{U}_{sc}$ *(hence* g *is also in* $\mathcal{U}_{sc}$*), then* $\pi(g) = \bigwedge_{\gamma} \pi(g_{\gamma})$.

(ii) If $g = \bigvee_{\gamma} g_{\gamma}$, *where* $\{g_{\gamma}\}_{\gamma \in \Gamma}$ *is a subset of* $\mathcal{L}_{sc}$ *(hence* g *is also in* $\mathcal{L}_{sc}$*), then* $\pi(g) = \bigvee_{\gamma} \pi(g_{\gamma})$.

S. Kaplan, *The bidual of C(X)* I, North-Holland Mathematics Studies 101, Amsterdam, 1985, p. 384.

Now we can proved (B), that is, for every $\widehat{f} \in C(X)^\delta$ we have

$$\bigvee\{g \in C(X) : g \leq \widehat{f}\} = \widehat{f} = \bigwedge\{g \in C(X) : g \geq \widehat{f}\}.$$

Indeed,

$$\begin{aligned} \widehat{f} &= \widehat{\underline{f}} = \pi(\underline{f}) = \pi(\ell(\underline{f})) = \pi\left(\bigvee\{g \in C(X) : g \leq \underline{f}\}\right) = \\ &= \bigvee\{g \in C(X) : g \leq \pi(\underline{f})\} = \bigvee\{g \in C(X) : g \leq \widehat{f}\}. \end{aligned}$$

and

$$\begin{aligned} \widehat{f} &= \widehat{\overline{f}} = \pi(\overline{f}) = \pi(u(\overline{f})) = \pi\left(\bigwedge\{g \in C(X) : g \geq \overline{f}\}\right) = \\ &= \bigvee\{g \in C(X) : g \leq \pi(\overline{f})\} = \bigvee\{g \in C(X) : g \leq \widehat{f}\}. \end{aligned}$$

Let us prove (A).

Theorem

$C(X)^\delta$ is **Dedekind complete**.

Proof. Let $\{\widehat{f_\gamma}\}$ be a subset of $C(X)^\delta$, which is bounded above by $\widehat{h}$, that is,

$$\widehat{f_\gamma} \leq \widehat{h}, \quad \text{for all } \gamma.$$

We can assume that $f_\gamma, h \in \mathcal{NL}_{sc} \Rightarrow f_\gamma = \ell u(f_\gamma)$ and $h = \ell u(h)$. Then

$$f_\gamma = \ell u(f_\gamma) \leq u(h), \text{ for all } \gamma.$$

So there exists $\bigvee f_\gamma$ in $\mathcal{B}(X)$ and $\bigvee f_\gamma \leq u(h)$. Define

$$f := \ell u \left(\bigvee f_\gamma \right).$$

Then $\bigvee \widehat{f_\gamma} = \widehat{f}$. Indeed,

(a) $f_\gamma = \ell u(f_\gamma) \leq \ell u \left(\bigvee f_\gamma \right) = f \Rightarrow \widehat{f_\gamma} \leq \widehat{f}$, for all γ.

(b) $f = \ell u \left(\bigvee f_\gamma \right) \leq \ell u(h) = h \Rightarrow \widehat{f} \leq \widehat{h}$.

- We have a proof of the Dedekind completion of $C(X)$, which used only the theory of Riesz spaces.

$$C(X)^{\delta} = \mathcal{C}_d(X)/\mathcal{R}a$$

$$\begin{aligned} C(X)^{\delta} &= \mathcal{N}\mathcal{U}_{sc}(X)/\mathcal{R}a = \mathcal{N}\mathcal{L}_{sc}(X)/\mathcal{R}a = \\ &= \mathcal{U}_{sc}(X)/\mathcal{R}a = \mathcal{L}_{sc}(X)/\mathcal{R}a = \mathcal{Q}(X)/\mathcal{R}a \end{aligned}$$

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- We have a proof of the Dedekind completion of $C(X)$, which used only the theory of Riesz spaces.
- The proof can be used in more general settings, if the operators ℓ and u can be defined.
- This proof can also be used for an alternative proof (without cuts) that an Archemedean Riesz space has a Dedekind completion.

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R. P. Dilworth

1950 - Dilworth introduced the **normal semicontinuous functions**, as functional analogous of normal subsets (cuts) of $C_b(X)$, and proved that for a **completely regular** space X there exists an isomorphism $C_b(X)^\delta \cong \mathcal{NU}_{sc}(X)$ only for lattice structures.
The set $\mathcal{NU}_{sc}(X)$ can be organized as a Riesz space with the operations:

$$\begin{aligned} f \overline{\oplus} g &= u\ell(f+g), \quad \lambda \overline{\odot} f = u\ell(\lambda f) = \begin{cases} \lambda f, & \lambda \geq 0, \\ \lambda \ell(f) & \lambda < 0. \end{cases}, \\ f \bigvee_{\mathcal{NU}} g &= f \vee g, \quad f \bigwedge_{\mathcal{NU}} g = u\ell(f \wedge g). \end{aligned}$$

R. P. Dilworth, *The normal completion of the lattice of continuous functions*, Trans. Amer. Math. Soc. 68 (1950), 427–438.

Alfred Horn

1953 - Horn proved a similar result as Dilworth, but for **unbounded** functions.
First he developed a general theory for the Dedekind completion of a subset C of a Dedekind complete lattice B.
Then he applied his theory for $B = \{f : X \longrightarrow \overline{\mathbb{R}}\}$ and $C = C(X, \overline{\mathbb{R}})$.
Horn proved that for a **completely regular** space X the following lattice isomorphism holds.
$C(X, \overline{\mathbb{R}})^{\delta} \cong \mathcal{N}\mathcal{L}_{sc}(X, \overline{\mathbb{R}})$
$C(X)^{\delta} = \mathcal{N}\mathcal{L}_{sc}^{cb}(X)$, where cb means functions which are C-bounded.
A function $f : X \longrightarrow \mathbb{R}$ is called C-**bounded** if there exists $g_1, g_2 \in C(X)$ such that $g_1 \leq f \leq g_2$, that is $f \in I(C(X))$.

A. Horn, *The normal completion of a subset of a complete lattice and lattices of continuous functions*, Pacific J. Math. 3 (1953), 137–152.

Kazumi Nakano and Tetsuya Shimogaki

1962 - K. Nakano and T. Shimogaki construct the Dedekind completion of $C(X)$ using **quasicontinuous functions**, as they were defined by Hidegorô Nakano in his book (1948, in Japanese) by the equality $u\ell(f) = u\ell u(f)$.
From the above theorems results:

1. A function $f \in \mathcal{B}(X)$ is **Nakano-quasicontinuous** if and only if is pointwise discontinuous.

Nakano, K., Shimogaki, T.; *A note on the cut extension of C-spaces*, Proc. Japan Acad., **8** (1962), 473-477.

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1. A function $f \in \mathcal{B}(X)$ is **Nakano-quasicontinuous** if and only if is pointwise discontinuous.
2. The Riesz ideal of the **rare** functions $\mathcal{R}a(X)$ coincides with the Riesz ideal N defined in the paper of Nakano and Shimogaki.

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2. The Riesz ideal of the **rare** functions $\mathcal{R}a(X)$ coincides with the Riesz ideal N defined in the paper of Nakano and Shimogaki.
3. The space $C_d(X)/\mathcal{R}a(X)$ is the space $C_q(X)/N$ of Nakano and Shimogaki.

Nakano, K., Shimogaki, T.; *A note on the cut extension of C-spaces*, Proc. Japan Acad., **8** (1962), 473-477.

Nakano, H.: *Measure Theory* II (in Japanese), Tokyo, 1948.

Samuel Kaplan

1964 - In a series of four papers and a book Kaplan studied intensively $C(X)$, his dual and his bidual. In this context he constructed the Dedekind completion of $C(X)$. From here I took the techniques for the proof.

S. Kaplan, *The second dual of the space of continuous functions*. IV, Trans. Amer. Math. Soc. 113 (1964), 512-546.

S. Kaplan, *The bidual of C(X)* I, North-Holland Mathematics Studies 101, Amsterdam, 1985.

S. N. Samborskii

2002 - Samborskii used **quasicontinuous** functions to construct a space called $S(X, \mathbb{R})$, which is nothing else that $C(X)^\delta$.
A function $f : X \longrightarrow \mathbb{R}$ is called **quasicontinuous** at $x \in X$ if for every $\varepsilon > 0$ and for every open set U containing x there exists a open set $G \subset U$ such that $|f(y) - f(x)| < \varepsilon$, for all $y \in G$.

$$f \text{ quasicontinuous} \Rightarrow \begin{cases} \ell(f) = \ell u(f) \\ u(f) = u\ell(f) \end{cases} \Rightarrow u\ell(f) = u\ell u(f) \Leftrightarrow \ell(\delta(f)) = 0$$

Samborskii, S. N., *Expansions of differential operators and nonsmoth solutions of differential equations*, Cybernetics and Systems Analysis, **38** (2002), 453-466.

Roumen Angelov

2004 - Anguelov constructed the Dedekind completion of $C(X)$ using **Hausdorff continuous interval-valued functions** (in the sens of Sendov), that is, the functions which associate at every point $x \in X$ the real closed interval $[\underline{f}(x), \overline{f}(x)]$,

$$\underline{\overline{f}} : X \longrightarrow \mathbb{IR}, \quad x \longrightarrow [\underline{f}(x), \overline{f}(x)],$$

and whose components forms a **regular pair**. If we denote by $\mathbb{H}(X)$ the set of all H-continuous functions on X, then the map

$$\Phi : C(X)^{\delta} \longrightarrow \mathbb{H}(X), \quad \Phi(\widehat{f}) = (\underline{f}, \overline{f})$$

is a Riesz isomorphism.

If we consider on $C(X)^{\delta}$ the **quotient norm** $\| \widehat{f} \| = \inf\{\|h\| : h \in \widehat{f}\}$ and on $\mathbb{H}(X)$ the **norm** $\|\underline{\overline{f}}\| = \max\{\| \underline{f} \|, \| \overline{f} \|\}$, Φ is an isometry.

R. Anguelov, *Dedekind order completion of C(X) by Hausdorff continuous functions*, Quaest. Math. 27 (2004), 153–169.

Sendov, B.: *Hausdorff Approximations*. Kluwer Academic Publishers, Dordrecht, The Netherlands, 1990.

The H-continuous functions **do not differ to much** from the usual **real-valued continuous** functions because they **assume interval values only on a set of first Baire category**. More precisely, the set

$$W_{\underline{f}} = \{x \in X \mid \overline{f}(x) - \underline{f}(x) > 0\}$$

is of the first Baire category. The function $\underline{f}$ has point values on the complementary set $D_{\underline{f}} = X \setminus W_{\underline{f}} = \{x \in X \mid \overline{f}(x) = \underline{f}(x)\}$, that is, $\underline{f}(x) = f(x)$, $x \in D_{\underline{f}}$, and $\underline{f} = f$ is a real valued continuous function on $D_{\underline{f}}$.. Therefore, a H-continuous function has the form

$$\underline{f}(x) = \begin{cases} f(x), & \text{if } x \in D_{\underline{f}}, \\ [\underline{f}(x), \overline{f}(x)], & \text{if } x \in W_{\underline{f}}. \end{cases}$$

Dedekind completion of $C(X)$ by cuts

Let X be a compact space and let (A, B) be a **cut** of $C(X)$. This means a pair of nonempty subsets of $C(X)$ such that

$$A^u = B \quad \text{and} \quad B^l = A.$$

Here A^u denotes the set of all **upper bounds** of A and B^l the set of all **lower bounds** of B.
Since A is bounded above and B is bounded below there exist the functions

$$\underline{f} = \sup A \quad \text{and} \quad \overline{f} = \inf B,$$

where $\sup A$ and $\inf B$ are computed pointwisely. Note that $\underline{f}$ is lower semicontinuous, $\overline{f}$ is upper semicontinuous, $\underline{f} \leq \overline{f}$, and they form a regular pair.

Theorem

Let X be a compact topological space.
Then every cut of $C(X)$ corresponds to a H-continuous interval-valued function, and conversely, every H-continuous interval-valued function corresponds to a cut of $C(X)$.

$$(A, B) \Longleftrightarrow [\underline{f}, \overline{f}]$$

More precisely, if (A, B) is a cut of $C(X)$ then the interval-valued function which corresponds is $\underline{\overline{f}} = [\underline{f}, \overline{f}]$, where $\underline{f} = \sup A$ and $\overline{f} = \inf B$.
Conversely, if $\underline{\overline{f}} = [\underline{f}, \overline{f}]$ is a H-continuous interval-valued function then the cut of $C(X)$ which corresponds is (A, B), where

$$A = \{g \in C(X) : g \leq \overline{f}\} \quad \text{and} \quad B = \{g \in C(X) : g \geq \underline{f}\}$$

R. P. Dilworth, *The normal completion of the lattice of continuous functions*, Trans. Amer. Math. Soc. 68 (1950), 427–438.

A. Horn, *The normal completion of a subset of a complete lattice and lattices of continuous functions*, Pacific J. Math. 3 (1953), 137–152.

S. Kaplan, *The second dual of the space of continuous functions*. IV, Trans. Amer. Math. Soc. 113 (1964), 512-546.

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W. A. J. Luxemburg and A. C. Zaanen, *Riesz Spaces*, vol. I, North-Holland Math. Library, North-Holland, Amsterdam-London, 1971.

H. M. MacNeille, *Partially Ordered Sets*, Trans. Amer. Math. Soc. 42 (1937), 416–460.

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Thank you for choosing this lecture room!