# On semigroups of nonnegative functions and positive operators

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Positivity VII, Leiden, July 2013

Joint work with Heydar Radjavi (Univ. of Waterloo, Canada).

# Entry-wise boundedness

Given a set $K \subseteq \mathbb{R}$, we write $M_n(K)$ for the set of all $n \times n$ matrices with entries in $K$.

**Theorem (Gessesse, Popov, Radjavi, Spinu, Tcaciuc, Troitsky)**

*Let $r > 1$ and $\mathscr{S}$ be a (multiplicative) semigroup in $M_n([0, r])$. Then there exists a diagonal matrix $D = \mathrm{diag}(d_1, \ldots, d_n)$ with $d_i \in [\frac{1}{r}, r]$ such that $D^{-1} \mathscr{S} D \subseteq M_n([0, 1])$.*

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A set $\mathscr{S}$ of nonnegative functions on $X \times X$, closed under a given associative operation $*$, is called a *matrix-like semigroup* if

$$(f * g)(x,z) \geq f(x,y)\, g(y,z)$$

for all $f$, $g \in \mathscr{S}$ and $x$, $y$, $z \in X$.

**Theorem**

*Let $M \geq 1$ be a real number and let $\mathscr{S}$ be a matrix-like semigroup of functions from $X \times X$ to $[0,M]$. Then there exists a function $d : X \mapsto [\frac{1}{M}, M]$ such that*

$$f(x,y) \leq \frac{d(x)}{d(y)}$$

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## Theorem

*Let $X$ be an arbitrary set, and let $f : X \times X \mapsto [0,\infty)$ be a function such that*

$$C_f(x,y) = \sup\{f(x,x_1)f(x_1,x_2)f(x_2,x_3)\cdots f(x_k,y) :$$

$$k \in \mathbb{N} \cup \{0\},\ x_1,\ldots,x_k \in X\} < \infty$$

*for all $x$ and $y$ in $X$.*

*(a) If there is a point $y_0 \in X$ such that $C_f(x,y_0) > 0$ for all $x \in X$, then there exists a function $d : X \mapsto (0,\infty)$ such that*

$$f(x,y) \le C_f(x,y) \le \frac{d(x)}{d(y)} \qquad (1)$$

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A semigroup $\mathscr{S} \subset M_n([0,\infty))$ is said to be *indecomposable (or irreducible)* if, for every $i$, $j$ in $\{1,\dots,n\}$, there exists $S \in \mathscr{S}$ with $S_{ij} > 0$. Equivalently: No permutation of the standard basis reduces the semigroup $\mathscr{S}$ to the block form

$$\begin{bmatrix} * & * \\ 0 & * \end{bmatrix}.$$

Or equivalently: $\mathscr{S}$ has no common non-trivial proper invariant ideals (i.e., subspaces spanned by a subset of the standard basis).

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*Let $\mathscr{S}$ be an indecomposable semigroup in $M_n([0,\infty))$ such that the set $\{S_{ij} : S \in \mathscr{S}\}$ is bounded for some pair $(i,j)$. Then there exists a diagonal matrix $D$ with positive diagonal entries such that $D^{-1}\mathscr{S}D \subseteq M_n([0,1])$.*

The indecomposability assumption cannot be omitted:

$$\mathscr{S} = \left\{ \begin{bmatrix} 1 & r \\ 0 & 1 \end{bmatrix} : r \in [0,\infty) \right\}.$$

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# Binary diagonals

A matrix is said to have a *binary diagonal* if its diagonal entries all come from the set $\{0,1\}$. Furthermore, a matrix is *binary* if its entries come from the set $\{0,1\}$.

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*Let $\mathscr{S}$ be an indecomposable semigroup of functions from $X \times X$ to $[0,\infty)$, where the multiplication of $f$ and $g$ in $\mathscr{S}$ is defined by*

$$(f * g)(x,y) = \sum_{z \in X} f(x,z)g(z,y).$$

*(Here the finiteness of the sum of nonnegative numbers is part of the hypothesis.) If every function $f \in \mathscr{S}$ has a binary diagonal, then there exists a function $d : X \mapsto (0,\infty)$ such that*

$$\frac{f(x,y)d(y)}{d(x)} \in \{0,1\}$$

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# Sketch of the proof:

Clearly, we may assume that $\mathscr{S}$ is maximal with respect to the inclusion. Then $\mathscr{S}$ contains the characteristic function of the diagonal of $X \times X$, which of course acts as an identity with respect to $*$.

Step 1: $\mathscr{S}$ contains the characteristic function of $\{(u,u)\}$ for each $u \in X$.

By Theorem above, there exists a function $d : X \mapsto (0,\infty)$ such that

$$f(x,y) \leq \frac{d(x)}{d(y)}$$

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# Finite diagonals and finite traces

A set $\mathscr{S}$ of complex matrices is said to have *finite diagonals* if all the diagonal entries of all the matrices in $\mathscr{S}$ come from a finite set, and it is called *self-adjoint* if for each $T \in \mathscr{S}$ we have $T^* \in \mathscr{S}$. Here $T^*$ is just the conjugate transpose of $T$.

**Theorem (Popov, Radjavi, Williamson)**

*Suppose that a semigroup $\mathscr{S}$ of nonnegative matrices has finite diagonals. If $\mathscr{S}$ is self-adjoint, then it is finite. Moreover, nonzero entries of matrices in $\mathscr{S}$ are of the form $\sqrt{\xi\eta}$, where $\xi$ and $\eta$ are diagonal values of some matrices in $\mathscr{S}$.*

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For a set $\mathscr{S}$ of operators on the Hilbert space $l^2$, we denote by $\mathscr{S}_+$ the set of all positive semidefinite operators in $\mathscr{S}$.

**Theorem**

*Let $\mathscr{S}$ be a self-adjoint semigroup of positive operators on the Banach lattice $l^2$. Suppose that either:*
*(i) $\mathscr{S}$ consists of trace-class operators and the set $\{\mathrm{tr}\, S : S \in \mathscr{S}_+\}$ is finite, or*
*(ii) for each $i \in \mathbb{N}$ the set $\{S_{ii} : S \in \mathscr{S}_+\}$ is finite.*
*Then the nonzero entries of $S \in \mathscr{S}$ are of the form $\sqrt{\xi \eta}$, where $\xi$ and $\eta$ are the diagonal entries of the projections $SS^*$ and $S^*S$, respectively.*

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Condition (i) of Theorem above does not imply condition (ii).

Let $c = \frac{1}{\sqrt{2}}$ and $f = (c, c^2, c^3, \ldots) \in l^2$. Then $\|f\| = 1$. For a positive integer $m$, let $g_m$ denote the vector obtained from $f$ by annihilating alternate segments of length $2^m$, i.e.,

$$g_m = (\underbrace{c, c^2, \ldots, c^{2^m}}_{2^m}, \underbrace{0, \ldots, 0}_{2^m}, \underbrace{c^{2^{m+1}+1}, c^{2^{m+1}+2}, \ldots, c^{3 \cdot 2^m}}_{2^m}, \underbrace{0, \ldots, 0}_{2^m}, \ldots),$$

and let $h_m = f - g_m$. Then the operator

$$Q_m = \frac{g_m g_m^*}{\|g_m\|^2} + \frac{h_m h_m^*}{\|h_m\|^2}$$

is a projection on $l^2$ of rank two. Define also the rank-one projection $P = f f^*$. Then the set

$$\mathscr{S} = \{P, Q_1, Q_2, Q_3, \ldots\}$$

is a semigroup, and, for each $i$, the set of all $(i, i)$ slots of members in $\mathscr{S}_+ = \mathscr{S}$ is infinite.

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