Factorization Theorems of Arendt Type for Additive Monotone Mappings

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Assume that $X = (X, \mathcal{A})$ is a measurable space and $\nu, \mu$ are measures defined on $X$. The Radon-Nikodym theorem says that $\nu$ is absolutely continuous with respect to $\mu$ (we write $\nu \ll \mu$) if and only if there exists a measurable function $g : X \to [0, +\infty)$ such that

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\int f \, d\nu = \int (f \cdot g) \, d\mu
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for all $f \in L^1(\nu)$. 
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Let $Y$ be a Banach space.

Assume that $(X, \mathcal{A})$ is a measurable space, $\mu$ is a finite vector measure having values in $Y$ and $\nu$ is a countably additive vector measure of bounded variation such that $|\nu| \ll \mu$.

We say that a Banach space $Y$ has the Radon-Nikodym property if there exist a $\mu$-integrable function $g: X \rightarrow Y$ such that:

$$\nu(E) = \int_E g \, d\mu, \quad E \in \mathcal{A}$$

Every reflexive Banach space has the Radon-Nikodym property. There are spaces which do not have the Radon-Nikodym property, e.g. $c_0$, $L^1(\Omega)$, $C(\Omega)$, $L^\infty(\Omega)$. 
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Put:

\[ S(f) := \int f \, d\nu, \quad T(h) := \int h \, d\mu, \quad \pi(f)(x) := f(x) \cdot g(x). \]

Note that \( S \) and \( T \) are positive operators and \( \pi \) is an orthomorphism, i.e. \( \pi \) is an order bounded linear mapping such that \( f \perp g \) implies \( \pi f \perp g \).

The assertion of the Radon-Nikodym theorem:

\[ \int f \, d\nu = \int (f \cdot g) \, d\mu \]

can be rewritten as follows: \( S = T \circ \pi \).
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Results of Maharam and the Luxemburg-Schep theorem


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Let $F$ and $G$ be two Riesz spaces and let $T : F \to G$ be a positive linear operator.

Then $T$ is said to have Maharam property if for all $f \in F$ and for all $g \in G$ such that $f \geq 0$ and $0 \leq g \leq Tf$ there exists some $f_1 \in F$ such that $0 \leq f_1 \leq f$ and $Tf_1 = g$. 
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Luxemburg-Schep theorem says that if Riesz spaces $F$ and $G$ are Dedekind complete and operator $T : F \to G$ is order continuous then the Maharam property of $T$ is equivalent to the following fact:

For every operator $S : F \to G$ such that $0 \leq S \leq T$ there exists an orthomorphism $\pi$ of $F$ such that $0 \leq \pi \leq I$ and $S = T \circ \pi$.

This is an operator version of the assertion of the Radon-Nikodym theorem.

The dual theorem: conditions for factorization $S = \pi \circ T$. 
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Factorization Theorems
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Theorem (Arendt)

Let $E$ be a Dedekind complete Riesz space, $F, G$ be Riesz spaces and $V : F \to G$ be a Riesz homomorphism. Then, given a positive linear mapping $S : G \to E$, every positive linear mapping $T : F \to E$ which satisfies $T \leq S \circ V$ admits a factorization

$$T = S_1 \circ V,$$

where $S_1 : G \to E$ is a linear mapping such that $0 \leq S_1 \leq S$. 

\[ \begin{array}{ccc}
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Theorem (Arendt)

Let $E$, $F$ and $G$ be Banach lattices with $G$ having an order-continuous norm and let $U : G \rightarrow F$ be an interval preserving positive linear mapping. Then, given a positive linear mapping $S : E \rightarrow G$, every positive linear mapping $T : E \rightarrow F$ which satisfies $T \leq U \circ S$ admits a factorization $T = U \circ S_1$, where $S_1 : E \rightarrow G$ is a linear mapping such that $0 \leq S_1 \leq S$. 
Some definitions

Let $\Phi : X \rightarrow \text{End}(G)$ be a representation of a semigroup $X$ in the semigroup $\text{End}(G)$ of endomorphisms of a group $G$. We will write $\Phi_s$ instead of $\Phi(s)$ for $s \in X$. Therefore:

$$\Phi_{st} = \Phi_s \circ \Phi_t, \quad s, t \in X.$$

If $X$ is a group, then we also have

$$\Phi_{s^{-1}} = (\Phi_s)^{-1}, \quad s \in X;$$

in particular every $\Phi_s$ is an invertible map.

A group with a lattice order compatible with its algebraic structure is called $\ell$-group.

A map $f : G \rightarrow F$ between $\ell$-groups is called monotone if

$$x \leq y \implies f(x) \leq f(y)$$

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for all $x, y \in G$ and $f$ is called $\Phi$-invariant if $f \circ \Phi_s = f$ for all $s \in X$. 
Assume that $E$ is a Dedekind complete Riesz space and $F$ and $G$ are Abelian $\ell$-groups. Further, denote by $\text{End}^+(G)$ the semigroup of all monotone endomorphisms of $G$. Moreover let $X$ be a right-amenable semigroup and let $\Phi: X \to \text{End}^+(G)$ be a representation of $X$. 

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Factorization Theorems
Let $V : F \to G$ be an $\ell$-group homomorphism such that $\Phi_s \circ V = V$ for all $s \in X$. Given an additive monotone and $\Phi$-invariant mapping $S : G \to E$, every additive monotone mapping $T : F \to E$ such that $T \leq S \circ V$ admits a factorization $T = S_1 \circ V,$

where $S_1 : G \to E$ is an additive and $\Phi$-invariant mapping such that $0 \leq S_1 \leq S.$
Result 2

Assume that $G$ is a Dedekind complete Riesz space and $E$ and $F$ are Abelian $\ell$-groups. Further, assume that $X$ is a right-amenable group and $\Phi: X \rightarrow \text{End}^+(E)$ is a representation of $X$ in the set of all monotone endomorphisms of $E$. 
Theorem

Let $U: G \to F$ be an injective $\ell$-group homomorphism. Given an additive monotone and $\Phi$-invariant mapping $S: E \to G$, every additive monotone and $\Phi$-invariant mapping $T: E \to F$ such that $T \leq U \circ S$ admits a factorization $T = U \circ S_1$,

where $S_1: E \to G$ is an additive and $\Phi$-invariant map such that $0 \leq S_1 \leq S$. 
Assume that $E$, $F$ and $G$ are Riesz spaces with $G$ having an order-continuous norm. Further, assume that $X$ is a right-amenable semigroup and $\Phi: X \to \mathcal{L}_p(G)$ is a representation of $X$ in the set $\mathcal{L}_p(G)$ of all positive linear self-mappings of $G$. 
Result 3

**Theorem**

Let $U: G \to F$ be an interval preserving and $\Phi$-invariant positive linear mapping. Given a positive linear mapping $S: E \to G$ such that $\Phi_s \circ S = S$ for all $s \in X$, every positive linear mapping $T: E \to F$ such that $T \leq U \circ S$ admits a factorization $T = U \circ S_1$,

where $S_1: E \to G$ is a linear map such that $0 \leq S_1 \leq S$ and $\Phi_s \circ S_1 = S_1$ for all $s \in X$. 
Assume that $E$, $F$, and $G$ are Riesz spaces with $E$ having an order-continuous norm. Further, assume that $X$ is a right-amenable semigroup and $\Phi : X \to \mathcal{L}_p(E)$ is a representation of $X$ in the set of all positive linear self-mappings of $E$. 
Theorem

Let \( V : F \to G \) be an interval preserving positive and injective linear mapping. Given a positive linear mapping \( S : G \to E \) such that \( \Phi_s \circ S = S \) for all \( s \in X \), every positive linear mapping \( T : F \to E \) such that \( T \leq S \circ V \) and \( \Phi_s \circ T = T \) for all \( s \in X \) admits a factorization \( T = S_1 \circ V \),

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\begin{array}{c}
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Thank you for your attention!!!