Disjointly homogeneous spaces: some bits and pieces

Julio Flores
University Rey Juan Carlos

Positivity and its applications, VII edition.
Leiden 2013

On the occasion of Prof. Zaanen's centenial

J.w.w. Hernández, Spinu, Tradacete and Troitsky

DH spaces

- Compactness properties of powers of $T : E \to E$

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  Key property:

  Every two normalized disjoint sequences in E are equivalent. In fact, every normalized disjoint sequence is equivalent to the unit basis of l_p or c_0.

DH spaces

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It characterizes L^p or c_0.

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Definition

A Banach lattice E is *DISJOINTLY HOMOGENEOUS* (***DH***) if for every pair (x_n), (y_n) of normalized disjoint sequences in E, there exists some subsequence (n_k) such that $(x_{n_k}) \approx (y_{n_k})$.

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Definition

E is p-DISJOINTLY HOMOGENEOUS, $1 \leq p < \infty$, (resp. ∞-DH) if every normalized disjoint sequence in E has some subsequence equivalent to the unit vector basis of ℓ_p (respectively c_0).

DH spaces

Remark

- *p-DH* $\Longrightarrow$ *DH. Are they equivalent?*

DH spaces

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- *p-DH* $\Longrightarrow$ *DH. Are they equivalent?*
- *It is a lattice property*
  *($L^p[0,1]$, $1 < p < \infty$, $p \neq 2$, with the lattice structure stemming from the inconditional Haar basis **is not** DH.)*

Examples of DH spaces: function spaces on $[0, 1]$

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- Lorentz spaces $L_{p,q}[0,1]$ and $\Lambda(\omega,q)[0,1]$, $1 \leq q < \infty$.

Theorem (Carothers, Dilworth, 87; Johnson, Tzafriri, 75)

Every normalized pairwise disjoint sequence in $L_{p,q}[0,1]$ (resp. $\Lambda(\omega,q)[0,1]$) has some subsequence equivalent to the unit basis of l_q, whose span is complemented $(1 \leq q < \infty)$.

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Proposition

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$(L_{p,\infty}[0,1])_o$ is ∞-DH (Novikov, Semenov, Tokarev, 79)

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$L_{p,\infty}[0,1]$, $1 < p < \infty$, is NOT disjointly homogeneous.

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$L_{p,\infty}[0,1]$ contains disjoint sequences spanning l_p (Kalton, 85).

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$\varphi : [0, \infty) \to [0, \infty)$ Orlicz function

$$E^\infty_{\varphi,s} = \overline{\left\{F \in C(0, 1) \;:\; F(\cdot) = \frac{\varphi(r\cdot)}{\varphi(r)} \;:\; r \geq s\right\}}$$

$(E^\infty_{\varphi,t} \subset E^\infty_{\varphi,s},\; t > s)$

$$E^\infty_\varphi \doteqdot \bigcap_{s>0} E^\infty_{\varphi,s} \subset C(0, 1), \quad C^\infty_\varphi = \overline{conv} E^\infty_\varphi$$

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Theorem (FHST, 2012)

$$L^\varphi[0,1]\ DH \iff E^\infty_\varphi\cong\{F\}$$

In such case $L^\varphi[0,1]$ is p-DH for some $1\leq p\leq\infty$ $(E^\infty_\varphi\cong\{t^p\},\ p\neq\infty)$.

Examples of DH spaces on $[0,1]$

$\Longrightarrow$)

Given $F \in E_{\varphi}^{\infty}$ there is $(t_n) \uparrow$ such that $\varphi(t_n) \geq 2^n$, and $|\frac{\varphi(t_n x)}{\varphi(t_n)} - F(x)| < 2^{-n}$ for all $x \in [0,1]$ and all $n \in \mathbb{N}$.

Take (A_n) disjoint with $\mu(A_n) = \varphi(t_n)^{-1}$ and $f_n = t_n \chi_{A_n}$

$$\int_0^1 \varphi(\sum \lambda_n f_n) < \infty \Longleftrightarrow \sum F(\lambda_n) < \infty$$

Given $G \in E_{\varphi}^{\infty}$

$$\int_0^1 \varphi(\sum \lambda_n g_n) < \infty \Longleftrightarrow \sum G(\lambda_n) < \infty$$

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Every normalized disjoint sequence in $L^\varphi(0,1)$ contains some subsequence equivalent to the unit vector basis of ℓ^G for some $G \in C^\infty_\varphi$.

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$(f_{n_k})_{L_\varphi} \approx (e_k)_{l^G}$ for some $G \in C_\varphi^\infty \approx \{F\}$, $(f_{n_k})_{L_\varphi} \approx (e_k)_{l^F}$

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In such case, L_φ is p-DH, $1 \leq p \leq \infty$

Examples of DH spaces on [0, 1]

Remark

$$\lim_{t\to\infty} \frac{t\varphi'(t)}{\varphi(t)} = p \implies E^{\infty}_{\varphi} \cong \{t^p\} \implies s(L^{\varphi}) = \sigma(L^{\varphi})$$

Examples of DH spaces: discrete spaces

- $l_p \ (1 \leq p \leq \infty)$, c_0.

Examples of DH spaces: discrete spaces

- l_p $(1 \leq p \leq \infty)$, c_0.
- Tsirelson space

Proposition (Casazza-Shura)

Let $y_n = \sum_{i=p_n+1}^{p_{n+1}} a_i t_i$ *be a normalized block basis sequence of* $\{t_n\}_1^\infty$. *Then* $[y_n]$ *is complemented in* T.

In addition, if $p_n < k_n \leq p_{n+1}$, $n = 1, 2 \ldots$ *are chosen, then*

$$(t_{k_n}) \approx (y_n).$$

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Every normalized disjoint sequence in T has a subsequence equivalent to some subsequence of $\{t_n\}_1^\infty$.

If $k_m < j_m$ for every m, then there is m_i such that $j_{m_{i-1}} < k_{m_i} < j_{m_i}$ for all i and

$$(t_{j_{m_i}}) \approx (t_{k_{m_i}})$$

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Corollary

$$p\text{-}DH \not\Leftrightarrow DH$$

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Stable spaces with a subsymmetric basis inducing the order cannot be DH unless they are l_p $(1 \leq p < \infty)$.

DH spaces: back to the beginning

Theorem (FHST, 2012)

Let E be a DH space and $T : E \to E$ a strictly singular operator.

- *If $\sigma(E) < \infty$ and E has an unconditional basis, then T^2 is compact. (Milman)*
- *If E is discrete and has a pairwise disjoint basis, then T is compact. (Pitt)*

Structure of DH spaces: some results

- Self-duality of the DH property.

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- Complementation of the span of disjoint sequences in DH spaces.

Structure of DH spaces: self-duality

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$$E \;\; \infty\text{-}DH \Longrightarrow E^* \;\; 1\text{-}DH$$

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Remark (FTT, 2009)

$$1 < p < \infty, \qquad \frac{1}{p} + \frac{1}{q} = 1$$

$$L_{p,1}[0,1] \;\; \textit{is 1-DH} \quad \textit{but} \quad L_{q,\infty}[0,1] \;\; \textit{is not DH.}$$

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Being DH **is not** a self-dual property.

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Where to look?

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$$L^{\varphi}[0,1] \text{ DH} \iff E_{\varphi}^{\infty} \cong \{t^p\} \iff E_{\varphi'}^{\infty} \cong \{t^q\} \iff L^{\varphi'}[0,1] \text{ DH}$$

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$$L_{p,q}[0,1] \text{ DH} \iff L_{p',q'}[0,1] \text{ DH}, \quad (1 < p < \infty, 1 \leq q < \infty)$$

Structure of DH spaces: self-duality

Take $\varphi : [0,\infty) \to [0,\infty)$ Orlicz function and $L^\varphi(0,\infty)$

$$E_\varphi(0,\infty) = \overline{\left\{ F \in C(0,1) \ : \ F(\cdot) \doteqdot \frac{\varphi(r\cdot)}{\varphi(r)} \ , \ \text{for some } r \in (0,\infty) \right\}}$$

$$C_\varphi(0,\infty) = \overline{\operatorname{conv}}\, E_\varphi(0,\infty)$$

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Theorem (FHSTT)

$L_{\varphi}(0,\infty)$ be a separable Orlicz space.

$$L^{\varphi}(0,\infty) \ p\text{-}DH \iff C_{\varphi}(0,\infty) \cong \{t^p\}, \ \textit{for some } 1 \le p < \infty$$

.

Structure of DH spaces: self-duality

Theorem (Nielsen, 1975)

If F is an Orlicz function equivalent to a function in $C_\varphi(0,\infty)$, then $L^\varphi(0,\infty)$ contains a lattice copy of ℓ^F.

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Theorem (Nielsen, 1975)

If F is an Orlicz function equivalent to a function in $C_\varphi(0,\infty)$, then $L^\varphi(0,\infty)$ contains a lattice copy of ℓ^F.
Every normalized disjoint sequence in $L^\varphi(0,\infty)$ contains some subsequence equivalent to the unit vector basis of ℓ^F for some $F \in C_\varphi(0,\infty)$.

Structure of DH spaces: self-duality

Theorem (FHSTT)

Let $1 < p < \infty$ and $\varphi(t)$ be an Orlicz function

- *It agrees with t^p on $[0, 1]$*
- $\varphi(t) \simeq t^p \log(1 + t)$ *on* $[1, \infty)$.

Then $L^\varphi(0, \infty)$ is a reflexive p-DH Banach lattice whose dual is not DH.

Structure of DH spaces: self-duality

Every function F in $C_\varphi(0,\infty)$ is a convex combination
$F = aF_1 + bF_2 + cF_3$ with
$F_1 \in C_{\varphi,1}, \quad F_2 \in C_\varphi^\infty, \quad F_3(t) = \int_1^\infty \frac{\varphi(st)}{\varphi(s)}\, d\mu(s)$

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$F_1 \in C_{\varphi,1}, \quad F_2 \in C_\varphi^\infty, \quad F_3(t) = \int_1^\infty \frac{\varphi(st)}{\varphi(s)}\, d\mu(s)$

$L^\varphi(0,\infty)^* = L^\psi(0,\infty)$

$\psi(t) = t^q, t \in [0,1]; \quad \psi(t) \approx \frac{t^q}{log^{q-1}(1+t)}, \; t \in [1,\infty).$

$\{t^q |logt|^\alpha\}_{0<\alpha<q-1} \subset C_\psi(0,\infty).$

Self-duality: positive results

Remark

An order continuous Banach lattice E with a weak unit can be represented as a dense order ideal in $L_1(\mu)$ for some probability measure μ, so we can consider E as a Köthe function space.

If (x_n) is a sequence in the order continuous Banach lattice E and $u = \sum_{n=1}^{\infty} \frac{x_n}{2^n \|x_n\|}$, then the closed ideal B_u generated by u is a projection band in E, $(x_n) \subset B_u$, and u is a weak unit in B_u; hence B_u can be represented as above.

Thus, every sequence in E is contained in some Köthe function space. Furthermore, B_u^* is a projection band in E^*, and if E (or E^*) is disjointly homogeneous then so is B_u (resp., B_u^*).

Self-duality: positive results

E^* DH
(x_n) and (y_n) disjoint in E.
(x_n^*) and (y_n^*) disjoint in E^* such that $x_n^*(x_m) = y_n^*(y_m) = \delta_{nm}$

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$$\begin{aligned}\Big\|\sum_{i=1}^m \alpha_i x_i\Big\| &= \sup\Big\{\Big|\langle\sum_{i=1}^m \alpha_i x_i, \sum_{i=1}^m \beta_i x_i^*\rangle\Big| \;:\; \Big\|\sum_{i=1}^m \beta_i x_i^*\Big\|_{[x_n]^*} \leq 1\Big\}\\ &= \sup\Big\{\Big|\sum_{i=1}^m \alpha_i\beta_i\Big| \;:\; \Big\|\sum_{i=1}^m \beta_i x_i^*\Big\|_{[x_n]^*} \leq 1\Big\}.\end{aligned}$$

Self-duality: positive results

$$\Big\|\sum_{i=1}^m \beta_i x_i^*\Big\|_{[x_n]^*} \leq \Big\|\sum_{i=1}^m \beta_i x_i^*\Big\|_{E^*} \underset{?}{\leq} M\Big\|\sum_{i=1}^m \beta_i x_i^*\Big\|_{[x_n]^*}$$

$$\begin{aligned}\Big\|\sum_{i=1}^m \alpha_i x_i\Big\| &\sim \sup\Big\{\Big|\sum_{i=1}^m \alpha_i\beta_i\Big| : \Big\|\sum_{i=1}^m \beta_i x_i^*\Big\|_{E^*} \leq 1\Big\} \\ &\sim \sup\Big\{\Big|\sum_{i=1}^m \alpha_i\beta_i\Big| : \Big\|\sum_{i=1}^m \beta_i y_i^*\Big\|_{E^*} \leq 1\Big\} \sim \Big\|\sum_{i=1}^m \alpha_i y_i\Big\|,\end{aligned}$$

Self-duality: positive results

Is there $S\colon [x_n]^* \to E^*$ bounded such that $Sx_m^* = x_m^*$?
(abuse of notation)

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Definition (FHSTT)

*A Banach lattice E has **property** $\mathfrak{P}$ if for every disjoint positive normalized sequence $(f_n) \subset E$ there exists an operator $T : E \to [f_n]$, such that some subsequence $(T^* f_{n_k}^*)$ is equivalent to a seminormalized disjoint sequence in E^* (here (f_n^*) denote the corresponding biorthogonal functionals in $[f_n]^*$).*

Self-duality: positive results

Lemma

Let E be a reflexive Banach lattice. The following are equivalent:

1. *For every disjoint positive normalized sequence* $(f_n) \subset E$ *there exists a positive operator* $T : E \to [f_n]$, *with* $\liminf_n \operatorname{dist}(f_n, T(B_E)) < 1$.
2. *For every disjoint positive normalized sequence* $(f_n) \subset E$ *there exists a positive operator* $T : E \to [f_n]$, *such that* $\|T^* f_n^*\| \nrightarrow 0$.
3. *E has property* $\mathfrak{P}$.

Self-duality: positive results

Lemma

X reflexive Banach space, (x_n) and (w_n^*) basic sequences in X and X^*, respectively, and $Tx = \sum_{n=1}^{\infty} w_n^*(x)x_n$. Then,

- $T^*x^* = \sum_{n=1}^{\infty} x^*(x_n)w_n^*$ *for each* $x^* \in X^*$.
- *If (y_n) and (z_n^*) are two basic sequences in X and X^*, respectively, such that $(x_n) \overset{C_1}{\approx} (y_n)$ and $(z_n^*) \overset{C_2}{\approx} (w_n^*)$, then $Sx := \sum_{n=1}^{\infty} z_n^*(x)y_n$ converges for every $x \in X$ and $\|S\| \le C_1 C_2 \|T\|$.*

Self-duality: positive results

Proposition (FHSTT)

Let E be a reflexive Banach lattice with property $\mathfrak{P}$.
E^* DH $\Longrightarrow$ E is DH.
E^* p-DH $\Longrightarrow$ E is q-DH $\quad (1 < p < \infty,\ \frac{1}{p} + \frac{1}{q} = 1)$.

Self-duality: positive results

(x_n), (y_n) disjoint sequences in E. Take $x_n^*(x_m) = \delta_{mn}$.

$(T^* x_{n_k}^*) \approx (g_k^*)$, $\quad (g_k^*)$ disjoint.

(h_k^*) normalized disjoint sequence in E^* such that $h_m^*(y_{n_k}) = \delta_{mk}$

$(g_{k_j}^*) \approx (h_{k_j}^*)$ as E^* is DH.

$$(T^* x_{n_{k_j}}^*) \overset{C}{\approx} (h_{k_j}^*)$$

Self-duality: positive results

$$
\begin{aligned}
\Big\|\sum_{j=1}^{m}\alpha_j x_{n_{k_j}}\Big\| &= \sup\Big\{\Big|\langle\sum_{j=1}^{m}\alpha_j x_{n_{k_j}}, \sum_{j=1}^{m}\beta_j x^*_{n_{k_j}}\rangle\Big| : \Big\|\sum_{j=1}^{m}\beta_j x^*_{n_{k_j}}\Big\|_{[x_n]^*} \leq 1\Big\}\\
&= \sup\Big\{\Big|\sum_{j=1}^{m}\alpha_j\beta_j\Big| : \Big\|\sum_{j=1}^{m}\beta_j x^*_{n_{k_j}}\Big\|_{[x_n]^*} \leq 1\Big\}\\
&\leq \sup\Big\{\Big|\sum_{j=1}^{m}\alpha_j\beta_j\Big| : \Big\|\sum_{j=1}^{m}\beta_j T^* x^*_{n_{k_j}}\Big\|_{E^*} \leq \|T^*\|\Big\}\\
&\leq C\|T^*\|\sup\Big\{\Big|\sum_{j=1}^{m}\alpha_j\beta_j\Big| : \Big\|\sum_{j=1}^{m}\beta_j h^*_{k_j}\Big\| \leq 1\Big\}
\end{aligned}
$$

Self-duality: positive results

$$
\begin{aligned}
&= \quad C\|T^*\| \sup\Big\{\Big|\langle\sum_{j=1}^{m}\alpha_j y_{n_{k_j}}, \sum_{j=1}^{m}\beta_j h^*_{k_j}\rangle\Big| \;:\; \|\sum_{j=1}^{m}\beta_j h^*_{k_j}\| \leq 1\Big\} \\
&\leq \quad C\|T^*\|\Big\|\sum_{j=1}^{m}\alpha_j y_{n_{k_j}}\Big\|.
\end{aligned}
$$

Self-duality: positive results

Start all over again with $(y_{n_{k_j}})$ and $(x_{n_{k_j}})$

$$\|\sum_{i=1}^{m} \alpha_i y_{n_{k_{j_i}}}\| \leq C' \|\sum_{i=1}^{m} \alpha_i x_{n_{k_{j_i}}}\|$$

$$(x_{n_{k_{j_i}}}) \approx (y_{n_{k_{j_i}}})$$

Self-duality: positive results

Spaces with property $\mathfrak{P}$?

Corollary

Let E be a reflexive Banach lattice satisfying an upper p-estimate. If E^* is q-DH ($\frac{1}{p} + \frac{1}{q} = 1$), then E is p-DH.

(f_n) arbitrary positive disjoint sequence in E.
(x_n^*) positive disjoint sequence in E^*

$$J : l_q \to E^* \quad Je_n = x_n^* \quad \text{lattice isomorphism.}$$

$$S : l_p \to [f_n], \quad Se_n = f_n.$$

$$T : E \xrightarrow{J^*} l_p \xrightarrow{S} [f_n]$$

$$T^* f_n^* = x_n^*$$

Self-duality: positive results

Corollary

If E is a reflexive p-DH space and it satisfies a lower p-estimate, for some $1 < p < \infty$, then E^* is q-DH ($\frac{1}{p} + \frac{1}{q} = 1$).

Structure of DH spaces: complemented disjoint sequences

L_p spaces, Lorentz spaces $\Lambda(W, q),\ q < \infty$. Tsirelson's space...satisfy that every disjoint positive sequence has a subsequence whose span is complemented by a positive projection. This motivates the following

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Definition

A Banach lattice E is called ***disjointly complemented*** (DC) if every disjoint sequence (x_n) has some subsequence whose span is complemented in E.

"(x_n) is complemented in E" if the span $[x_n]$ is complemented in E.

Structure of DH spaces: complemented disjoint sequences

Question

$$DH \Longrightarrow DC?$$

Complemented disjoint sequences: some remarks

Let's start with something simpler

Question

Does every reflexive Banach lattice contain a complemented positive disjoint sequence?

Complemented disjoint sequences: some remarks

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Does every reflexive Banach lattice contain a complemented positive disjoint sequence?

- E with a Schauder basis such that the order is compatible with the basis.

Complemented disjoint sequences: some remarks

Let’s start with something simpler

Question

Does every reflexive Banach lattice contain a complemented positive disjoint sequence?

- E with a Schauder basis such that the order is compatible with the basis.
- Rearrangement invariant spaces (averaging projection).

Complemented disjoint sequences: some remarks

Proposition

E reflexive contains a complemented positive disjoint sequence if and only if E^* does.

Complemented disjoint sequences: some remarks

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E reflexive contains a complemented positive disjoint sequence if and only if E^* does.

Remark

If P is a positive projection onto the span of a disjoint sequence $(x_n) \subset E_+$, and (x_n^*) denote the biorthogonal functionals, then, in general, the sequence $(P^* x_n^*)$ need not be disjoint in E^*

Complemented disjoint sequences: some remarks

Example

$E = \mathbb{R}^3$ and let

$$P = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

$$P^* e_1^* = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \quad \text{and} \quad P^* e_2^* = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}.$$

Nevertheless,

Complemented disjoint sequences: some remarks

Proposition

Let E be reflexive and (f_n) a positive disjoint sequence complemented in E by the projection $R: E \to E$.

Then there exists a positive disjoint sequence (g_n^*) in E^* with $\langle g_n^*, f_m \rangle = \delta_{n,m}$ such that

$$Px = \sum_{n=1}^{\infty} g_n^*(x) f_n$$

defines a positive projection onto $[f_n]$ with $\|P\| \leq \|R\|$. ($P^* f_n^* = g_n^*$)

Complemented disjoint sequences: some remarks

Corollary

Given a positive disjoint sequence (e_n) in a reflexive Banach lattice E, the following are equivalent:

- *The subspace $[e_n]$ is complemented in E.*
- *There exists a disjoint positive sequence (w_n^*) in E^* with $\langle w_n^*, e_m \rangle = \delta_{nm}$ such that $\sum_{n=1}^{\infty} w_n^*(x) e_n$ converges for each $x \in E$. In this case, the map $P \colon x \mapsto \sum_{n=1}^{\infty} w_n^*(x) e_n$ defines a positive projection from E onto $[e_n]$.*

Complemented disjoint sequences: some remarks

Corollary

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Corollary

E reflexive contains a complemented positive disjoint sequence if and only if E^* does.

Complemented disjoint sequences: some remarks

Proposition

Let E be a DH reflexive Banach lattice. TFAE:

- *E has property* $\mathfrak{P}$
- *E contains a complemented positive disjoint sequence.*

Complemented disjoint sequences: some remarks

Lemma

X reflexive Banach space, (x_n) and (w_n^*) basic sequences in X and X^*, respectively, and $Tx = \sum_{n=1}^{\infty} w_n^*(x)x_n$. Then,

- *$T^*x^* = \sum_{n=1}^{\infty} x^*(x_n)w_n^*$ for each $x^* \in X^*$.*
- *If (y_n) and (z_n^*) are two basic sequences in X and X^*, respectively, such that $(x_n) \overset{C_1}{\approx} (y_n)$ and $(z_n^*) \overset{C_2}{\approx} (w_n^*)$, then $Sx := \sum_{n=1}^{\infty} z_n^*(x)y_n$ converges for every $x \in X$ and $\|S\| \leq C_1C_2\|T\|$.*

Complemented disjoint sequences: some remarks

Proof of the proposition

$\Longrightarrow$)

$(f_n) \subset E_+$ disjoint. $T^* f^*_{n_k} \approx g^*_k$, (g^*_k) disjoint.

$g^*_k(g_l) = \delta_{kl}$, $\quad (g_l) \subset E_+$ disjoint. We may assume $(g_k) \approx f_{n_k}$

$$x \to \sum_{k=1}^{\infty} T^* f^*_{n_k}(x) f_{n_k}, \quad x \to \sum_{k=1}^{\infty} g^*_k(x) g_k$$

are bounded.

Complemented disjoint sequences: some remarks

$\Longleftarrow$)
Let (e_n) be a complemented positive disjoint sequence and (f_n) disjoint in E.
Consider (e_n^*) and $Px = \sum e_n^*(x)e_n$

$Re_n = f_n$ is isomorphism $((e_n) \approx (f_n))$

$$T = RP : E \to [e_n] \to [f_n]$$

$$Tx = \sum e_n^*(x)f_n, \quad T^*x^* = \sum x^*(f_n)e_n^*$$

$T^*f_n^* = e_n^*$

DH and DC spaces: non-reflexive case

Remark

Let E be a non-reflexive DH space; then E is either 1-DH or ∞-DH.

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DH and DC spaces: non-reflexive case

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If E is a separable Banach lattice and ∞-DH then it is DC.

Proposition

If E is a 1-DH Banach lattice, then E is DC.

Corollary

If E is a separable non-reflexive Banach lattice which is DH, then E is DC.

DH and DC spaces: reflexive case

Proposition

Let E be a reflexive Banach lattice containing a complemented positive disjoint sequence.

- *If E^* is DH, then*

$$E \text{ DH} \iff E \text{ has } \mathfrak{P} \iff E \text{ DC}$$

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- *If E and E^* are DH, then E is DC.*

DH and DC spaces: reflexive case

Proposition

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- *If E is DH, then*

$$E^* \ DH \iff E^* \ has \ \mathfrak{P} \iff E^* \ DC$$

- *If E and E^* are DH, then E is DC.*
- *Let E be a p-DH Banach lattice which is p-convex with $1 < p < \infty$. Then E is DC.*

DH spaces: summary

- DH spaces generalize a good property of L^p-spaces (Milman, Pitt).

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DH spaces: summary

- DH spaces generalize a good property of L^p-spaces (Milman, Pitt).
- DH is not a self dual property.
- There are partial duality results when E has $\mathfrak{P}$ property.
- E reflexive , $s(E) \geq p$. E^* q-DH $\Longrightarrow$ E p-DH.
- Every sequence of pairwise disjoint vectors in E DH has some subsequence which is complemented (up to being untrue...).