On p-convergent operators on Banach lattices

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Outline of the talk:

1. Recall the definition of the concept "p-convergent operator on Banach spaces" from the paper [4] (J. Castillo & F. Sánchez, Dunford-Pettis-like Properties of Continuous Vector Function Spaces, *Revista Matemática de la Universidad Complutense de Madrid* **6**(1) (1993)) and discuss some properties of p-convergent operators.
2. Introduce the so called DP^* property of order p on Banach spaces, modelled on the concept of DP^* property studied in the paper [3](H. Carrión, P. Galindo & M.L. Lourenço, A stronger Dunford-Pettis property, *Studia Mathematica* **3** (2008)) and relate this with the "Dunford-Pettis property of order p" studied in [4].
3. Consider some applications to homogeneous polynomials and holomorphic functions on Banach spaces.
4. Discuss some introductory results about p-convergent operators on Banach lattices.

1 p-Convergent operators on Banach spaces

A sequence (x_n) in a Banach space X is called weakly p-summable (for $1 \leq p < \infty$) if

$$\sum_{i=1}^{\infty} |\langle x_i, x^* \rangle|^p < \infty$$

for all $x^* \in X^*$.
In the case when $p = \infty$ we agree (for the purpose of this lecture) to consider the weak null sequences, i.e. sequences which are in $c_0^{weak}(X)$.
From [4] (Castillo & Sánchez) we recall the following definition:

Definition 1.1 *Let* $1 \leq p \leq \infty$. *We call* $T \in \mathcal{L}(X, Y)$ *p-convergent if it transforms weakly-p-summable sequences into norm-null sequences.*

The class of p-convergent operators from X into Y, is denoted by $\mathcal{C}_p(X, Y)$.

$\mathcal{C}_\infty(X,Y)$ is the class of completely continuous (or Dunford-Pettis) operators and $\mathcal{C}_1(X,Y)$ is the class of unconditionally summing operators (those transforming weakly-1-summable sequences into summable ones). Obviously, $\mathcal{C}_q \subset C_p$ when $p < q$. Also, $\mathcal{C}_p$ is a closed (in the operator norm) injective operator ideal.

Our discussion will make use of several definitions from the paper [4](Castillo & Sánchez):

Definition 1.2 *Let $1 \leq p \leq \infty$. A sequence (x_n) in a Banach space X is said to be weakly-p-convergent with limit $x \in X$ if the sequence $(x_n - x)$ is weakly-p-summable.*

The weakly-∞-convergent sequences are the weakly convergent sequences.

Definition 1.3 *Let $1 \leq p \leq \infty$. A bounded set K in a Banach space is relatively weakly-p-compact (resp. weakly-p-compact) if every sequence in K has a weakly-p-convergent subsequence (resp. with limit in K).*

The relatively weakly-∞-compact sets are simply the relatively weakly compact sets.

Remark 1.4 *It is well known, for instance from the paper [2] (S.I. Ansari, On Banach spaces* Y *for which* $B(C(\Omega), Y) = K(C(\Omega), Y)$*, Pac. J. Math.* **169***(2) (1995)), that for* $1 < p < \infty$*, and* $1/p + 1/p^* = 1$ *we have that* $\mathcal{L}(\ell_{p^*}, X) = K(\ell_{p^*}, X)$ *iff all weak-p-summable sequences in* X *are norm null, i.e. iff* $id_X \in C_p$*.*

The well-known notion of "weak Dunford-Pettis operator" motivates the following definition of "weak p-convergent operator":

Definition 1.5 *An operator* T *from a Banach space* X *into a Banach space* Y *is called weak p-convergent if* $(y_n^*(Tx_n))$ *converges to 0 for every sequence* $(x_n) \in \ell_p^{weak}(X)$ *and every weak null sequence* (y_n^*) *in* Y^**.*

Clearly each p-convergent operator is weak p-convergent. Also,

Proposition 1.6 *Let X, Y be Banach spaces and $T \in \mathcal{L}(X, Y)$. T is weak p-convergent if and only if for all Banach spaces Z and all weakly compact operators $S : Y \to Z$, the operator ST is p-convergent.*

From Proposition 1.6, taking $S = id_Y$, it follows that

Corollary 1.7 *If Y is a reflexive Banach space, then each weak p-convergent operator $T : X \to Y$ is p-convergent.*

A subset L of a Banach space X is called *limited* if $weak^*$ null sequences in the (continuous) dual space X^* of X converge uniformly on L.

In the paper [3] (Carrión, Galindo & Lourenço) the authors define and discuss the following variant of the well-known Dunford-Pettis property:

Definition 1.8 *X is said to have the DP^*P when all weakly compact sets in X are limited, i.e. when each sequence $(x_n^*) \subset X^*$ which converges weak* to 0, converges uniformly to 0 on all weakly compact sets in X.*

In the paper [7] (E. D. Zeekoei & JHF, *DP^*-properties of order p on Banach spaces*, *Quaestiones Mathematicae*, to appear) the following property on Banach spaces is introduced:

Definition 1.9 *A Banach space X is said to have the DP^*-property of order p (for $1 \leq p \leq \infty$) if all weakly-p-compact sets in X are limited. In short, we write DP^*P_p (for DP^*-property of order p).*

If $q > p$, then DP^*P_q implies DP^*P_p; $DP^*P = DP^*P_\infty$ implies DP^*P_p for all $1 \leq p < \infty$.

Theorem 1.10 *Let $1 \leq p \leq \infty$. The Banach space X has the DP^*P_p if and only if $\langle x_n, x_n^* \rangle \to 0$ as $n \to \infty$ for all $\sigma(X^*, X)$ null sequences (x_n^*) in X^* and all weakly-p-summable sequences (x_n) in X.*

If a weakly null sequence (x_n) in a separable Banach space X is not norm null then, by passing to subsequences if necessary, there is a weak* null sequence (x_n^*) in the metrisable topological space $(B_{X^*}, \sigma(X^*, X))$ such that $|\langle x_n, x_n^* \rangle| \nrightarrow 0$. Since weakly-$p$-summable sequences in X are weakly null, it therefore follows that if in a separable Banach space X there exists a weakly-p-summable sequence (x_n) which is not norm null, then by Theorem 1.10 the space X does not have the DP^*P_p.

Thus, by Remark 1.4, if a separable Banach space X has the DP^*P_p (for $1 < p < \infty$), then $K(\ell_{p^*}, X) = \mathcal{L}(\ell_{p^*}, X)$.

In the paper [4](Castillo & Sánchez), a Banach space X is said to have the DPP_p if the inclusion $\mathcal{W}(X,Y) \subseteq \mathcal{C}_p(X,Y)$ holds for all Banach spaces Y. Here $\mathcal{W}(X,Y)$ denotes the family of all weakly compact operators from X to Y. It is also proved in the paper [4] that a Banach space X has the DPP_p if and only if $\langle x_n, x_n^* \rangle \to 0$ as $n \to \infty$ for all weakly null sequences (x_n^*) in X^* and all weakly-p-summable sequences (x_n) in X. Again, If $q > p$, then DPP_q implies DPP_p; $DPP = DPP_\infty$ implies DPP_p for all $1 \leq p < \infty$.

Clearly, the DP^*P_p always implies the DPP_p and when a Banach space is a Grothendieck space (hence also if it is a reflexive space) it has the DP^*P_p iff it has the DPP_p.

For instance, if $1 < r < \infty$, then ℓ_r has the DPP_p for all $p < r^*$. Being reflexive, it has the DP^*P_p for all $p < r^*$, but not the DPP (hence aslo not the DP^*P).

The (non-reflexive) $C(K)$-spaces and L_1-spaces have the DPP and therefore they also have the DPP_p for all p. However, the following example shows that the DPP_p does not always imply the DP^*P_p:

Example 1.11 *Let (Ω, Σ, μ) be some probability space. The space $L_1(\mu)$ has the DPP (by the Dunford-Pettis Theorem) and thus also has the DPP_p for all $1 \leq p \leq \infty$. By Remark 1.4 every weakly-2-summable sequence in $L_1(\mu)$ would be a norm null sequence if and only if each bounded linear operator from the sequence space ℓ_2 to $L_1(\mu)$ were compact. This is impossible, since for instance we know that ℓ_2 embeds isometrically in $L_1(\mu)$. Thus, there has to be a weakly-2-summable sequence which is not norm null. Since $L_1(\mu)$ is separable, this implies that $L_1(\mu)$ does not have the DP^*P_2.*

The previous example serves to separate the class of spaces with the DPP_p from the class of spaces with the DP^*P_p. On the other hand, the following example shows the existence of a space with the DP^*P_p which does not have the DP^*P_q (for some $q > p$):

Example 1.12 *Let $1 < r < 2$ and consider any $1 < p < r$. Since $p < r^*$, it follows that ℓ_r has the DP^*P_p. Since the continuous embedding $\ell_p \hookrightarrow \ell_r$ is not compact and since it follows again by Remark 1.4 that every weakly-p^*-summable sequence in ℓ_r would be a norm null sequence if and only if each bounded linear operator from ℓ_p to ℓ_r would be compact, the same argument as in Example 1.11 yields that the space ℓ_r does not have the DP^*P_q where $q = p^* > r > p$.*

Proposition 1.13 *Let $1 \leq p < \infty$. The Banach space X has the DP^*P_p if and only if every operator $T \in \mathcal{L}(X, c_0)$ is p-convergent, i.e. if and only if $\mathcal{L}(X, c_0) = \mathcal{C}_p(X, c_0)$.*

Following is a result that relates the properties DPP_p and DP^*P_p:

Theorem 1.14 *The Banach space X has the DP^*P_p if and only if it has the DPP_p and every quotient mapping $q : X \to c_0$ is p-convergent.*

2 Applications to homogeneous polynomials and holomorphic functions on Banach spaces.

Definition 2.1 *Let $1 \leq p < \infty$. A function f from a normed space X into a normed space Y is called p-convergent if it maps weakly-p-convergent sequences onto norm convergent sequences.*

In case of $p = \infty$ in the above definition, f is a completely continuous function. The focus is on the p-convergent property for n-homogeneous polynomials. The case $p = \infty$ was covered in the paper [3](Carrión, Galindo & Lourenço).

Recall that if E,F are vector spaces over $\mathbb{C}$ and $\Delta_n : E \to E^n$ is the diagonal mapping defined by $\Delta_n x = (x, x, \ldots, x)$, then a mapping $P : E \to F$ is called an *n-homogeneous polynomial* from E to F if there exists a n-linear $L \in \mathcal{L}_a(^nE; F)$ from $E^n \to F$ such that $P = L \circ \Delta_n$; i.e. $P(x) = L(x, x, \ldots, x)$ for all $x \in E$. When considering normed spaces E and F, we agree to use $\mathcal{P}(^nE; F)$, for the space of continuous n-homogeneous polynomials from E into F.

Proposition 2.2 *Let X, Y be Banach spaces and assume Y contains an isomorphic copy of c_0. If every $T \in \mathcal{L}(X, Y)$ is p-convergent, then X has the DP^*P_p. In this case, every polynomial $P \in \mathcal{P}(^nX, Y)$ is a p-convergent function for all $n \in \mathbb{N}$.*

Using some (adjusted) results from the paper [3] (Carrión, Galindo & Lourenço) one verifies that:

Proposition 2.3 *Let X be a Banach space with the DP^*P_p and let $(P_n) \subset \mathcal{P}(^kX, \mathbb{C})$ such that $P_n \to P \in \mathcal{P}(^kX, \mathbb{C})$ pointwise. Then $P_n \to P$ uniformly on weakly-p-compact subsets of X and for each weakly-p-summable sequence (x_n) we have $P_n(x_n) \underset{\infty}{\overset{n}{\to}} 0$.*

Denote the class of all p-convergent k-homogeneous polynomials from X into Y by $\mathbb{P}_{pc}(^kX, Y)$.

Proposition 2.4 *The following assertions are equivalent:*

(i) X has the DP^*P_p.

(ii) Every operator $T : X \to c_0$ is p-convergent.

(iii) For all integers k, each polynomial $P \in \mathcal{P}(^kX, c_0)$ is p-convergent.

(iv) For some integer k, each polynomial $P \in \mathcal{P}(^kX, c_0)$ is p-convergent.

Recall the definition of holomorphic function:

Definition 2.5 *Let U be an open subset of X. A mapping $f : U \to F$ is said to be holomorphic (or analytic) if for each $a \in U$ there exists a ball $B(a,r) \subset U$ and a sequence of polynomials $P_m \in \mathcal{P}(^m X; Y)$ such that*

$$f(x) = \sum_{m=0}^{\infty} P_m(x-a),$$

uniformly on $B(a,r)$. The vector space of all holomorphic mappings from U into Y is denoted by $\mathcal{H}(U;F)$.

If $f \in \mathcal{H}(X,Y)$ is p-convergent, then it is bounded on all weakly-p-compact subsets of X. Recall that Banach spaces whose limited sets are relatively compact are called *Gelfand-Philips spaces.*

Proposition 2.6 *If X has the DP^*P_p and Y is a Gelfand-Phillips space, then every $P \in \mathcal{P}(^n X, Y)$ is p-convergent. Furthermore, each $f \in \mathcal{H}(X,Y)$ which is bounded on weakly-p-compact sets, is weakly continuous on them.*

Recall that a subset L of a Banach space X is called *bounding* if every $f \in \mathcal{H}(X, \mathbb{C})$ is bounded on L. Bounding sets in a Banach space X are limited (a result from [5] (B. Josefson, A Banach space containing non-trivial limited sets but no non-trivial bounding sets, *Israel J. Math* (1990)). So, if all weakly-p-compact subsets of X are bounding, then X has the DP^*P_p and since $\mathbb{C}$ is a Gelfand-Phillips space, it follows from Proposition 2.6 that in this case each $f \in \mathcal{H}(X, \mathbb{C})$ has to be p-convergent. Moreover, we have:

Proposition 2.7 *All weakly-p-compact subsets of a Banach space X are bounding iff each $f \in \mathcal{H}(X, \mathbb{C})$ is p-convergent.*

The following proposition describes yet another characterisation of Banach spaces with the DP^*P_p. The proof of the case $p = \infty$ is considered in [3] (Carrión, Galindo & Lourenço). The same techniques apply in our setting of the DP^*P_p, where $1 \leq p < \infty$.

Proposition 2.8 *A Banach space X has the DP^*P_p if, and only if, every symmetric bilinear separately compact map $X \times X \to c_0$ is p-convergent.*

3 p-Convergent operators on Banach lattices

Lemma 3.1 *Let Ω be a compact Hausdorff topological space and let $1 \leq p < \infty$. For a weak p-summable sequence (f_j) in $C(\Omega)$ we have:*

(1) $(f_j(t))_j \in \ell_p$ *for all* $t \in \Omega$.

(2) $\sup_{t \in \Omega} \|(f_j(t))_j\|_{\ell_p} < \infty$.

Using the Riesz Representation Theorem and the above Lemma 3.1, one verifies that:

Lemma 3.2 *Let $1 \leq p < \infty$. If (f_j) is a weak p-summable sequence in $C(\Omega)$ then so is the sequence $(|f_j|)$.*

For any AM-space with unit, being lattice isometric to some $C(\Omega)$-space, it follows from the above lemmas that:

Proposition 3.3 *Let* $1 \leq p < \infty$*. Let* E *be an* AM*-space with unit. Then* $(|x_i|) \in \ell_p^w(E)$ *for each* $(x_i) \in \ell_p^w(E)$*.*

Definition 3.4 *We say a Banach lattice* E *is weak p-consistent (for* $1 \leq p < \infty$*) if it follows from* $(x_i) \in \ell_p^w(E)$ *that* $(|x_i|) \in \ell_p^w(E)$*.*

Proposition 3.5 *Let* E *be a weak p-consistent Banach lattice and* F *any Banach lattice. If* $S, T : E \to F$ *are positive operators satisfying* $0 \leq S \leq T$ *and* T *is p-convergent, then likewise* S *is p-convergent.*

When the target space is an AL-space, then we have the following easy characterisation of a p-convergent operator:

Proposition 3.6 *Let E be a Banach lattice and let F be an AL-space. Then the following are equivalent:*

(1) T is p-convergent.

(2) $|Tx_n| \to 0$ as $n \to \infty$ weakly in F for all $(x_i) \in \ell_p^w(E)$.

Following are some introductory results in a study which is currently mostly motivated by the paper [6] (M. Moussa & K. Bouras, About positive weak Dunford–Pettis operators on Banach lattices, *Journal of Mathematical Analysis and Applications*, **381** (2011)), although a series of papers by W. Wnuk on this topic will certainly also play important role in this study.

Definition 3.7 *Let E be a Banach lattice and Y be a Banach space. We call an operator $T : E \to Y$ almost p-convergent if for every sequence $(x_n) \subset E^+$ such that $(x_n) \in \ell_p^{weak}(E)$, we have $\|Tx_n\| \xrightarrow[\infty]{n} 0$.*

Remarks 3.8 *(a) By definition, each p-convergent operator is almost p-convergent.*

(b) Weak p-convergent operators differ from almost p-convergent operators:
For instance, c_0 has DPP_p (since c_0 has DPP), thus id_{c_0} is weak p-convergent (by Proposition 1.6). However, since $(e_n) \in \ell_p^{weak}(c_0)$, (e_n) is a disjoint sequence in c_0^+ and $\|e_n\| \nrightarrow 0$, it follows that id_{c_0} is not almost p-convergent.

Definition 3.9 *A Banach space E is said to have the positive Schur property of order p (briefly, E has SP_p^+) if each disjoint sequence $(x_n) \in \ell_p^{weak}(E)$ with positive terms, is norm convergent to 0.*

Theorem 3.10 *Let E be a Banach lattice. Then, the following statements are equivalent:*

(1) Each positive operator from E to ℓ_∞ is almost p-convergent.

(2) E has SP_p^+.

Theorem 3.11 *Let E and F be two Banach lattices such that F is a dual Banach lattice. Then the following assertions are equivalent:*

(1) Each positive weak p-convergent operator $T : E \to F$ is almost p-convergent.

(2) One of the following assertions is valid:

(a) E has SP_p^+.

(b) F is a KB-space.

Replacing the condition that F be a dual Banach lattice in Theorem 3.11 with the condition that F is Dedekind σ-complete, we obtain the following result:

Theorem 3.12 *Let E and F be two Banach lattices with F Dedekind σ-complete. If each positive weak p-convergent operator $T : E \to F$ is almost p-convergent,then one of the following assertions is valid:*

(1) E has SP_p^+.

(2) F has an order continuous norm.

References

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