Asymptotic Behavior of Semigroups of Kernel Operators

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Greiner's theorem

Theorem

Assume that

- E *Banach lattice with order continuous norm,*
- $\mathscr{T} = (T(t))_{t \geq 0}$ *positive, bdd, irreducible* C_0*-semigroup on* E*,*
- $T(t) \wedge T(s) > 0$ *for some* $t \neq s$*,*
- *Generator* A *satisfies* $\sigma_p(A) \cap i\mathbb{R} = \{0\}$

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Then there exist $0 < y \in \mathrm{Fix}(\mathscr{T})$ *and* $0 < y' \in \mathrm{Fix}(\mathscr{T}')$ *such that*

$$\lim_{t\to\infty} T(t)x = \langle y', x\rangle y \quad \text{for all } x \in E.$$

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Satisfied for kernel operators
under the assumptions of the theorem.

Kernel operators

Definition

Operator $T \in \mathscr{L}(L^p(\Omega, \mu))$ is a *kernel operator* if there exists a measurable $k : \Omega \times \Omega \to \mathbb{R}$ such that for every $f \in L^p(\Omega, \mu)$

$$k(x, \cdot)f(\cdot) \in L^1(\Omega, \mu)$$

and

$$Tf = \int_\Omega k(\,\cdot\,, y)f(y)\,\mathrm{d}\mu(y).$$

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On a Banach lattice E:

$T \in \mathscr{L}(E)$ is a *kernel operator* if $T \in (E' \otimes E)^{\perp\perp}$.

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Theorem (Axmann)

If T is positive and irreducible and dominates a kernel operator, then $T^n \wedge T^m > 0$ for some $n \neq m$.

D. Axmann, *Struktur-und Ergodentheorie irreduzibler Operatoren auf Banachverbänden*. PhD thesis, Tübingen, 1980.

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See also: W. Arendt, *Positive semigroups of kernel operators*. Positivity, 2008.

Generalizations

Theorem

Let $\mathscr{T} = (T(t))_{t \geq 0}$ positive, bdd, irreducible C_0-semigroup on E.

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Generalizations

Theorem

Let $\mathscr{T} = (T(t))_{t \geq 0}$ positive, bdd, irreducible C_0-semigroup on E.

- *If $T(t_0)$ dominates a compact or kernel operator, then $\sigma_p(A) \cap i\mathbb{R} \subseteq \{0\}$.*

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Let $\mathscr{T} = (T(t))_{t\geq 0}$ *positive, bdd, irreducible* C_0*-semigroup on* E*.*

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- *If* $\mathrm{Fix}(\mathscr{T}) \neq \{0\}$ *and* $T(t_0)$ *dominates a kernel operator, then*

$$\lim_{t\to\infty} T(t)x = \langle y', x\rangle y$$

for all $x \in E$*.*

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Discrete Version

Theorem

Let $T \in \mathscr{L}(E)$ positive, irreducible, power bdd and $\mathrm{Fix}(T) \neq \{0\}$.

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Let $T \in \mathscr{L}(E)$ *positive, irreducible, power bdd and* $\mathrm{Fix}(T) \neq \{0\}$*.*
If T *dominates a kernel operator, then there exist* $k \in \mathbb{N}$*,*
$0 < y' \in \mathrm{Fix}(T')$ *and* $0 < y \in \mathrm{Fix}(T)$ *such that*

$$\lim_{n\to\infty} T^{kn}x = \langle y', x\rangle y \quad \text{for all } x \in E.$$

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Let $T \in \mathscr{L}(E)$ *positive, expanding, power bdd and* $\mathrm{Fix}(T) \neq \{0\}$*.*
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Definition

A *(bdd) transition kernel* is $k : \Omega \times \mathscr{B}(\Omega) \to \mathbb{R}$ such that

- $k(\,\cdot\,, A)$ measurable function for all $A \in \mathscr{B}(\Omega)$,
- $k(x,\,\cdot\,)$ is a Borel measure for all $x \in \Omega$,
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Such operator T is called *weakly continuous*.

Weakly continuous and kernel operators

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Let $\mathscr{T} = (T(t))_{t\geq 0}$ *positive semigroup on* $B_b(\Omega)$ *such that*

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- *there exists strictly positive* $\mathscr{T}$*-invariant measure* μ.

M. G., R. Nittka, *A new proof of Doob's theorem*, J. Math. Anal. Appl., 2012

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Then equivalent:

(i) $T^*(t)$ *(weakly continuous) operator on* $\mathscr{M}(\Omega)$ *for all* $t > 0$

(ii) $T(t)$ *extends to a kernel operator on* $L^1(\Omega, \mu)$ *for all* $t > 0$

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Sublattice of weakly continuous operators

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The weakly continuous operators are a sublattice of $\mathscr{L}(\mathscr{M}(\Omega))$.

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- If $k : \Omega \times \mathscr{B}(\Omega) \to \mathbb{R}$ is a transition kernel, then $|k|$ is a transition kernel. Recall: $|k|(x, \cdot\,)$ is total variation of $k(x, \cdot\,)$.

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But: Weakly continuous operators are no ideal!

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Theorem (Doob)

Let $\mathcal{T} = (T(t))_{t\geq 0}$ be stochastically continuous Markovian semigroup on $\mathscr{M}(\Omega)$. Assume that

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Then

$$\lim_{t\to\infty} T(t)\nu = \nu(\Omega)\cdot\mu$$

in norm for all $\nu \in \mathscr{M}(\Omega)$.

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$T(s) \wedge T(t)$ is given by $k_s \wedge k_t > 0$, hence $T(t) \wedge T(s) > 0$.

Thank you for your attention!