On the moduli and characteristic of monotonicity in Orlicz-Lorentz function spaces

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Denote $X_+ = \{x \in X \colon x \geq 0\}$ and

$$S_+(X) = S(X) \cap X_+.$$

Definition

A Banach lattice X is said to be **strictly monotone** ($X \in$ (**SM**)), if for all $x, y \in X_+$ such that $y \leq x$ and $y \neq x$, we have $\|y\| < \|x\|$.

Equivalently: X is strictly monotone, if for all $y \in X_+$ and $x \in S_+(X)$ such that $y \leq x$ and $y \neq x$, we have $\|x - y\| < \|x\|$.

Definition

A Banach lattice X is said to be **uniformly monotone** ($X \in$ (**UM**)), if

$$\underset{0<\varepsilon<1}{\forall} \quad \underset{\delta(\varepsilon)\in(0,1)}{\exists} \quad \underset{0\leq y\leq x,\ \|x\|=1}{\forall} \left(\|y\| \geq \varepsilon\right) \Rightarrow \|x - y\| \leq 1 - \delta(\varepsilon). \quad (1)$$

Definition

Let X be a Banach lattice. The function $\delta_{m,X} : [0,1] \to [0,1]$ defined by

$$\delta_{m,X}(\varepsilon) = \inf\{1 - \|x - y\| : 0 \leq y \leq x, \|x\| = 1, \|y\| \geq \varepsilon\} \quad (2)$$

is said to be **the (lower) modulus of monotonicity** of X.

Remark

$$1)\ X \in (\mathbf{UM}) \Leftrightarrow \delta_{m,X}(\varepsilon) > 0 \text{ for every } \varepsilon \in (0,1].$$

$$2)\ X \in (\mathbf{SM}) \Leftrightarrow \delta_{m,X}(1) = 1.$$

Fact

Let us define for any $\varepsilon \in [0,1]$*:*

$$\delta_{m,X}^{S,\geq}(\varepsilon) = \inf\{1 - \|x - y\| : 0 \leq y \leq x, \|x\| = 1, \|y\| \geq \varepsilon\},$$

$$\delta_{m,X}^{S,=}(\varepsilon) = \inf\{1 - \|x - y\| : 0 \leq y \leq x, \|x\| = 1, \|y\| = \varepsilon\},$$

$$\delta_{m,X}^{B,\geq}(\varepsilon) = \inf\{1 - \|x - y\| : 0 \leq y \leq x, \|x\| \leq 1, \|y\| \geq \varepsilon\},$$

$$\delta_{m,X}^{B,=}(\varepsilon) = \inf\{1 - \|x - y\| : 0 \leq y \leq x, \|x\| \leq 1, \|y\| = \varepsilon\}.$$

Then

$$\delta_{m,X}^{S,\geq}(\varepsilon) = \delta_{m,X}^{S,=}(\varepsilon) = \delta_{m,X}^{B,\geq}(\varepsilon) = \delta_{m,X}^{B,=}(\varepsilon) \quad \forall\, \varepsilon \in [0,1].$$

Fact [Kurc, 1993]

The modulus of monotonicity $\delta_{m,X}(.)$ of a normed lattice X is:

- a convex function on the interval $[0, 1]$, which is continuous on the interval $[0, 1)$,
- a nondecreasing function on the interval $[0, 1]$.

Remark

The modulus of monotonicity $\delta_{m,X}(.)$ needn't be left continuous at the point 1.

Definition

Let X be a Banach lattice. The number $\varepsilon_{0,m}(X) \in [0,1]$ defined as

$$\sup\{\varepsilon \in [0,1] \colon \delta_{m,X}(\varepsilon) = 0\} \tag{3}$$

is said to be **the characteristic of monotonicity** of X.

Remark

$\varepsilon_{0,m}(X) = \sup\{\varepsilon \in [0,1] \colon \delta_{m,X}(\varepsilon) = 0\} = \inf\{\varepsilon \in [0,1] \colon \delta_{m,X}(\varepsilon) > 0\}$, *where* $\inf \emptyset := 1$.

Remark

$$X \in (\mathbf{UM}) \Leftrightarrow \varepsilon_{0,m}(X) = 0.$$

Theorem [Betiuk-Pilarska & Prus, 2008]

Suppose that X is a weakly orthogonal Banach lattice with $\varepsilon_{0,m}(X) < 1$. Then X has weak normal structure.

Definition

A Banach lattice X is said to be **weakly orthogonal** *if*

$$\liminf_{n\to\infty} \liminf_{m\to\infty} \| |x_n| \wedge |x_m| \| = 0$$

whenever (x_n) is a sequence in X which converges weakly to 0.

Theorem [Joint with Foralewski, Kaczmarek and Krbec, 2010]

For any Banach lattice X the following formula for the characteristics of monotonicity hold true:

$$\varepsilon_{0,m}(X) = \sup\{\limsup_{n\to\infty} \|x_n - y_n\| : \|x_n\| = 1, 0 \le y_n \le x_n \underset{n\in\mathbb{N}}{\forall}, \|y_n\| \to 1\}. \quad (4)$$

Collorary

In any finite dimensional Banach lattice X the characteristic of monotonicity is just the length of the longest order interval lying in the intersection of the unit sphere of X and X_+, i.e.

$$\begin{aligned} \varepsilon_{0,m}(X) &= \sup\{\|x-y\| : 0 \le y \le x, \|x\| = \|y\| = 1\} \quad (5) \\ &= \max\{\|x-y\| : 0 \le y \le x, \|x\| = \|y\| = 1\}. \end{aligned}$$

Theorem [Joint with Foralewski, Kaczmarek and Krbec, 2010]

For any Banach lattice X the following equality is true

$$\varepsilon_{0,m}(X) = 1 - \lim_{\varepsilon \to 1^-} \delta_{m,X}(\varepsilon). \tag{6}$$

Moreover,

$$\delta_{m,X}(1 - \delta_{m,X}(\varepsilon)) = 1 - \varepsilon \tag{7}$$

for arbitrary $\varepsilon \in (\varepsilon_{0,m}(X), 1]$ if $\varepsilon_{0,m}(X) < 1$ as well as also in the case when $\varepsilon = \varepsilon_{0,m}(X) = 1$.

Remark

In equality (6), $\lim_{\varepsilon \to 1^-} \delta_{m,X}(\varepsilon)$ cannot be replaced by $\delta_{m,X}(1)$. There are examples of Banach lattices X for which $\delta_{m,X}(\varepsilon) = 0$ for any $\varepsilon \in [0, 1)$ and $\delta_{m,X}(1) = 1$.

Example

For any Lorentz space $\Lambda_\omega = \{x \in L^0 \colon \|x\| = \int_0^\infty x^*(t)\omega(t)dt < \infty\}$ such that the weight function ω is not regular but $\int_0^\infty \omega(t)dt = \infty$ (for example $\omega(t) = \min(1, 1/t)$ for $t \in [0, \infty)$), we have

$$\delta_{m,\Lambda_\omega}(1^-) = 0 < 1 = \delta_{m,\Lambda_\omega}(1).$$

Corollary

For arbitrary Banach lattice X the following formulas hold true

$$\begin{aligned}\varepsilon_{0,m}(X) &= \lim_{\varepsilon\to 1^-}\left(\sup\{\|x-y\| : 0\le y\le x, \|x\|=1, \|y\|\ge\varepsilon\}\right)\\ &= \lim_{\varepsilon\to 1^-}\left(\sup\{\|x-y\| : 0\le y\le x, \|x\|=1, \|y\|=\varepsilon\}\right).\end{aligned}$$

Let us denote by:

- (T, Σ, μ) a positive, complete and $\sigma-$finite measure space,
- $L^0 = L^0(T, \Sigma, \mu)$ the space of all (equivalence classes of) real-valued and $\Sigma-$measurable functions defined on T,
- $E = (E, \leq, \|\cdot\|_E)$ denotes a Köthe space over the measure space (T, Σ, μ), that is E is a Banach subspace of L^0 which satisfies the following conditions:
  (i) If $|x| \leq |y|$, $y \in E$ and $x \in L^0$, then $x \in E$ and $\|x\|_E \leq \|y\|_E$.
  (ii) There exists a function $x \in E$ which is strictly positive $\mu-$a.e. in T.

Let us define for a Köthe space E the modulus $\widehat{\delta}_{m,E} : [0,1] \to [0,1]$ by the formula

$$\widehat{\delta}_{m,E}(\varepsilon) = \inf \left\{ 1 - \|x - x\chi_A\|_E : x \geq 0, \|x\|_E = 1, A \in \Sigma, \|x\chi_A\|_E \geq \varepsilon \right\}.$$

The characteristic of monotonicity $\widehat{\varepsilon}_{0,m}(E)$ corresponding to the modulus $\widehat{\delta}_{m,E}$ is defined by

$$\widehat{\varepsilon}_{0,m}(E) = \sup \left\{ \varepsilon \in [0,1] \colon \widehat{\delta}_{m,E}(\varepsilon) = 0 \right\} = \inf \left\{ \varepsilon \in [0,1] \colon \widehat{\delta}_{m,E}(\varepsilon) > 0 \right\}, \quad (8)$$

where $\inf \emptyset = 1$.

The modulus $\widehat{\delta}_{m,E}$ is nondecreasing with respect to $\varepsilon \in [0,1]$ and

$$\delta_{m,X}(\varepsilon) \leq \widehat{\delta}_{m,E}(\varepsilon) \leq \varepsilon \ \ \forall\, \varepsilon \in [0,1]. \tag{9}$$

It is easy to see that

$$\begin{aligned}
\widehat{\delta}_{m,E}(\varepsilon) &= \inf\left\{1 - \|x - x\chi_A\|_E : x \geq 0, \|x\|_E = 1, A \in \Sigma, \|x\chi_A\|_E = \varepsilon\right\} \\
&= 1 - \sup\left\{\|x - x\chi_A\|_E : x \geq 0, \|x\|_E = 1, A \in \Sigma, \|x\chi_A\|_E \geq \varepsilon\right\} \\
&= 1 - \sup\left\{\|x - x\chi_A\|_E : x \geq 0, \|x\|_E = 1, A \in \Sigma, \|x\chi_A\|_E = \varepsilon\right\}.
\end{aligned}$$

Proposition [Joint with Foralewski, Kaczmarek and Krbec, 2010]

For arbitrary Köthe space E the following formula holds true

$$\widehat{\varepsilon}_{0,m}(E) = \sup\left\{\limsup_{n\to\infty}\left\|x_n\chi_{A_n'}\right\|_E : (x_n) \subset S_+(E), (A_n) \subset \Sigma, \|x_n\chi_{A_n}\|_E \to 1\right\}.$$

Lemma (*)

If E is a Köthe space then for any positive ε and δ satisfying the condition $\varepsilon + \delta < 1$ the inequality $\delta_{m,E}(\varepsilon+\delta) \geq \delta\widehat{\delta}_{m,E}(\varepsilon)$ holds true.

Proof.

Let $\varepsilon, \delta \in (0,1)$ be such that $\varepsilon + \delta < 1$ and $\widehat{\delta}_{m,E}(\varepsilon) > 0$. Assume that $0 \leq y \leq x$, $\|x\|_E = 1$ and $\|y\|_E \geq \varepsilon + \delta$. Let us define

$$A = \{t \in T \colon y(t) < \delta x(t)\}.$$

Then $\|y\chi_A\|_E \leq \|\delta x\|_E = \delta$. Since $\varepsilon + \delta \leq \|y\|_E \leq \|y\chi_A\|_E + \|y\chi_{A'}\|_E$ we get that $\|y\chi_{A'}\|_E \geq \varepsilon$. Therefore

$$\begin{aligned} \|x-y\|_E &\leq \|x - y\chi_{A'}\|_E \leq \|x - \delta x\chi_{A'}\|_E \\ &= \|(1-\delta)x + \delta x - \delta x\chi_{A'}\|_E \\ &\leq (1-\delta)\|x\|_E + \delta\|x - x\chi_{A'}\|_E \\ &\leq (1-\delta) + \delta\left(1 - \widehat{\delta}_{m,E}(\varepsilon)\right) = 1 - \delta\widehat{\delta}_{m,E}(\varepsilon). \end{aligned}$$

Hence for all $0 \leq y \leq x$ such that $\|x\|_E = 1$, $\|y\|_E \geq \varepsilon + \delta$, we have that $1 - \|x-y\|_E \geq \delta\widehat{\delta}_{m,E}(\varepsilon)$, whence $\delta_{m,E}(\varepsilon+\delta) \geq \delta\widehat{\delta}_{m,E}(\varepsilon)$. In consequence, for any $0 < \varepsilon < \bar{\varepsilon}$ we have $\widehat{\delta}_{m,E}(\varepsilon) > 0 \Rightarrow \delta_{m,E}(\bar{\varepsilon}) > 0$. □

Theorem [Joint with Foralewski, Kaczmarek and Krbec, 2010]

For arbitrary Köthe space E we have the equality

$$\varepsilon_{0,m}(E) = \widehat{\varepsilon}_{0,m}(E).$$

Corollary

For arbitrary Köthe space X the following formulas are true

$$\begin{aligned}
\varepsilon_{0,m}(E) = \widehat{\varepsilon}_{0,m}(E) &= \lim_{\varepsilon\to 1^-} \sup\left\{\left\|x\chi_{A'}\right\|_E : x \in S_+(E), A \in \Sigma, \|x\chi_A\|_E \geq \varepsilon\right\} \\
&= \lim_{\varepsilon\to 1^-} \sup\left\{\left\|x\chi_{A'}\right\|_E : x \in S_+(E), A \in \Sigma, \|x\chi_A\|_E = \varepsilon\right\}.
\end{aligned}$$

Proof of the Theorem

Since $\delta_{m,E}(\varepsilon) \leq \widehat{\delta}_{m,E}(\varepsilon)$ for all $\varepsilon \in [0,1]$, we have

$$\widehat{\varepsilon}_{0,m}(E) \leq \varepsilon_{0,m}(E). \qquad (10)$$

In order to get the inequality $\widehat{\varepsilon}_{0,m}(E) \geq \varepsilon_{0,m}(E)$, we need to consider separately two cases; namely the case when $\varepsilon_{0,m}(E) < 1$ and the case when $\varepsilon_{0,m}(E) = 1$.

Case 1. Assume that $\varepsilon_{0,m}(E) < 1$. By virtue of inequality (10), we have $\widehat{\varepsilon}_{0,m}(E) < 1$ and $\widehat{\delta}_{m,E}(\varepsilon) > 0$ for all $\varepsilon \in (\widehat{\varepsilon}_{0,m}(E), 1)$. By Lemma (*), we have $\delta_{m,E}(\varepsilon_1) \geq (\varepsilon_1 - \varepsilon)\widehat{\delta}_{m,E}(\varepsilon) > 0$ for all ε and ε_1 such that $\widehat{\varepsilon}_{0,m}(E) < \varepsilon < \varepsilon_1 < 1$. Therefore, we obtained that $\delta_{m,E}(\varepsilon_1) > 0$ for any $\varepsilon_1 \in (\widehat{\varepsilon}_{0,m}(E), 1)$. Hence

$$\varepsilon_{0,m}(E) := \inf\{\varepsilon_1 : \delta_{m,E}(\varepsilon_1) > 0\} \leq \widehat{\varepsilon}_{0,m}(E).$$

Proof of the Theorem

Case 2. Assume now that $\varepsilon_{0,m}(E) = 1$. We will prove that $\widehat{\varepsilon}_{0,m}(E) = 1$. Assume for the contrary that $\widehat{\varepsilon}_{0,m}(E) < 1$. Then, similarly as in Case 1, we get that $\delta_{m,E}(\varepsilon_1) > 0$ for all $\varepsilon_1 \in (\widehat{\varepsilon}_{0,m}(E), 1)$, whence $\varepsilon_{0,m}(E) \leq \widehat{\varepsilon}_{0,m}(E) < 1$, a contradiction. Therefore $\varepsilon_{0,m}(E) = 1$ implies that $\widehat{\varepsilon}_{0,m}(E) = 1$.

Results in Orlicz-Lorentz function spaces

Let in the following $L^0 = L^0([0,\gamma))$ be the space of all (equivalence classes of) Lebesgue measurable real-valued functions defined on the interval $[0,\gamma)$, where $\gamma \leq \infty$. Denoting the Lebesgue measure by m, for any $x \in L^0$ we define its distribution function $\mu_x : [0,+\infty) \to [0,\gamma]$ by

$$\mu_x(\lambda) = m\{t \in [0,\gamma) : |x(t)| > \lambda\}$$

and its nonincreasing rearrangement $x^* : [0,\gamma) \to [0,\infty]$ as

$$x^*(t) = \inf\{\lambda \geq 0 : \mu_x(\lambda) \leq t\}$$

(under the convention $\inf \emptyset = \infty$). We say that two functions $x, y \in L^0$ are **equimeasurable** if $\mu_x(\lambda) = \mu_y(\lambda)$ for all $\lambda \geq 0$. It is obvious that equimeasurability of x and y gives the equality $x^* = y^*$.

Definition

Let (R_1, Σ_1, μ_1) and (R_2, Σ_2, μ_2) be complete σ-finite measure spaces. A map σ from R_1 into R_2 is called **a measure preserving transformation** *if for any Σ_2-measurable subset A from R_2, the set $\sigma^{-1}(A) = \{t \in R_1 : \sigma(t) \in A\}$ is a Σ_1-measurable subset of R_1 and $\mu_1(\sigma^{-1}(A)) = \mu_2(A)$.*

Remark

It is well known that a measure preserving transformation induces equimeasurability, that is, if σ is a measure preserving transformation, then x and $x \circ \sigma$ are equimeasurable functions. The converse is false.

Definition

A Köthe space E, where $E \subset L^0$, is called **a symmetric space** *if E is rearrangement invariant which means that if $x \in E$, $y \in L^0$ and $x^* = y^*$, then $y \in E$ and $\|x\| = \|y\|$.*

Definition

Let $\omega : [0, \gamma) \to R_+$ be a non-increasing and locally integrable function (not identically 0), called **a weight function**. *We say that* **a weight function ω is regular** *if there exists $\eta > 0$ such that*

$$\int_0^{2t} \omega(s)ds \geq (1+\eta) \int_0^t \omega(s)ds$$

for any $t \in [0, \gamma/2)$.

In the whole presentation Φ denotes **an Orlicz function**, that is, $\Phi : [-\infty, \infty] \to [0, \infty]$ (our definition is extended from R into R^e by assuming $\Phi(-\infty) = \Phi(\infty) = \infty$) and Φ is convex, even, vanishing and continuous at zero, left continuous on $(0, \infty)$ (that is, in particular, $\lim_{u \to (b(\Phi))^-} \Phi(u) = \Phi(b(\Phi))$, where

$$b(\Phi) = \sup \{u \geq 0 \colon \Phi(u) < \infty\})$$

and not identically equal to zero on $(-\infty, \infty)$.

Definition

We say that an Orlicz function Φ satisfies **condition Δ_2** *for all $u \in \mathbb{R}_+$* **(respectively, at infinity)** *if there is $K > 0$ such that the inequality $\Phi(2u) \leq K\Phi(u)$ holds for all $u \in \mathbb{R}$* **(respectively, for all $u \in \mathbb{R}$ satisfying $|u| \geq u_0$ with some $u_0 > 0$ such that $\Phi(u_0) < \infty$)**. *We write then $\Phi \in \Delta_2(\mathbb{R}_+)$ ($\Phi \in \Delta_2(\infty)$), respectively.*

In the following we will use the parameter $a(\Phi)$ for the Orlicz function Φ defined by

$$a(\Phi) := \sup\{u > 0 \colon \Phi(u) = 0\}.$$

Given any Orlicz function Φ and any non-increasing weight function ω, we define on L^0 the convex modular

$$\mathrm{I}_{\Phi,\omega}(x) = \int_0^\gamma \Phi(x^*(t))\omega(t)dt,$$

and the Orlicz-Lorentz space

$$\Lambda_{\Phi,\omega} = \Lambda_{\Phi,\omega}([0,\gamma)) = \{x \in L^0 : \mathrm{I}_{\Phi,\omega}(\lambda x) < \infty \quad for \quad some \quad \lambda > 0\},$$

which becomes a Banach symmetric space under the Luxemburg norm

$$\|x\|_{\Lambda_{\Phi,\omega}} = \inf\{\lambda > 0 : \mathrm{I}_{\Phi,\omega}(x/\lambda) \le 1\}.$$

In proofs of our results three lemmas that are presented below were applied.

Lemma [Kamińska, 1990]

Assume that $|x(t)| < |y(t)|$ for $t \in A \subset [0, \gamma)$, where $\mu(A) > 0$ and $|x(t)| \leq |y(t)|$ for m-a.e. $t \in [0, \gamma)$. If $\mu_x(\lambda) < \infty$ for any $\lambda > 0$, then $x^*(t) < y^*(t)$ for $t \in B$, where $B \subset [0, \gamma)$ has a positive measure.

Lemma

Let Φ be an Orlicz function with $a(\Phi) > 0$ and satisfying condition $\Delta_2(\infty)$ and let $c \in (a(\Phi), +\infty)$. Then for any $\varepsilon \in (0,1)$ there exists $\delta(\varepsilon) \in (0,1)$ such that if $x \in \Lambda_{\Phi,\omega}$, $|x(t)| \geq c$ for m-a.e. $t \in [0,\gamma)$ and $\mathrm{I}_{\Phi,\omega}(x) \leq \delta(\varepsilon)$, then $\|x\|_{\Lambda_{\Phi,\omega}} \leq \varepsilon$.

Lemma

Let $\gamma < \infty$ and $\Phi \in \Delta_2(\infty)$. Then for any $\varepsilon \in (0,1)$ there exists $p(\varepsilon) \in (0,1)$ such that $\|x\|_{\Lambda_{\Phi,\omega}} \leq 1 - p(\varepsilon)$ whenever $\mathrm{I}_{\Phi,\omega}(x_n) \leq 1 - \varepsilon$.

Theorem

Let $\gamma = \infty$. If the Orlicz function Φ satisfies condition $\Delta_2(\mathbb{R})$ and the weight function ω is regular, then $\varepsilon_{0,m}(\Lambda_{\Phi,\omega}) = 0$. In the opposite case, $\varepsilon_{0,m}(\Lambda_{\Phi,\omega}) = 1$.

Remark

It is worth noticing here that if $\Phi \in \Delta_2(\mathbb{R})$ and $\int_0^\infty \omega(t)dt = \infty$, then $\Lambda_{\Phi,\omega}$ is strictly monotone (even if ω is not regular), whence $\delta_{m,\Lambda_{\Phi,\omega}}(1) = 1$.

Theorem

Suppose that $0 < \gamma < \infty$, Φ *is an Orlicz function satisfying condition* $\Delta_2(\infty)$ *and define* $\gamma_0 := \sup\{t \in [0,\gamma) : \omega(t) > 0\}$, *where* ω *is a weight function on* $[0,\gamma)$. *Let us denote by* $u(\Phi,\omega)$ *the positive number satisfying the equality* $\Phi(u(\Phi,\omega)) \int_0^{\gamma_0} \omega(t)dt = 1$. *Moreover, in the case when* $\gamma_0 < \gamma$, *let us define the positive number* $v(\Phi,\omega)$ *by the formula* $\Phi(v(\Phi,\omega)) \int_0^{\gamma-\gamma_0} \omega(t)dt = 1$. *Then the following assertions are true:*

(i) *If* $\gamma_0 = \gamma$, *then* $\delta_{m,\Lambda_{\Phi,\omega}}(1) = 1 - \frac{a(\Phi)}{u(\Phi,\omega)}$.

(ii) *If* $\gamma_0 \in (\frac{1}{2}\gamma, \gamma)$, *then*

$$\delta_{m,\Lambda_{\Phi,\omega}}(1) = 1 - \max\left(\frac{a(\Phi)}{u(\Phi,\omega)}, \frac{u(\Phi,\omega)}{v(\Phi,\omega)}\right).$$

(iii) *If* $\gamma_0 \in (0, \frac{1}{2}\gamma]$, *then* $\delta_{m,\Lambda_{\Phi,\omega}}(1) = 0$.

The example presented below shows how the value $\delta_{m,\Lambda_{\Phi,\omega}}(1)$ can be varying in dependence on γ_0, γ and $a(\Phi)$.

Example 3

Let $\gamma > 0$, $\Phi(u) = \max\{0, u - 1\}$ for any $u \geq 0$, $\omega(t) = 1$ for any $t \in [0, \min(1, \gamma))$ and $\omega(t) = 0$ for any $t \in [1, \gamma)$ whenever $\gamma > 1$. Then $a(\Phi) = 1$ and $u(\Phi, \omega) = \max((\frac{1}{\gamma} + 1), 2)$. By the above Theorem, statements ((*iii*) and (*i*)), we get the equalities $\delta_{m,\Lambda_{\Phi,\omega}}(1) = 0$ for $\gamma \geq 2$ and $\delta_{m,\Lambda_{\Phi,\omega}}(1) = \frac{1}{\gamma+1}$ when $\gamma \leq 1$. Assume now that $\gamma \in (1, 2)$. Then $v(\Phi, \omega) = \frac{\gamma}{\gamma-1}$ and, by the above Theorem, statement (*ii*), we have $\delta_{m,\Lambda_{\Phi,\omega}}(1) = \frac{1}{2}$ for $\gamma \in (1, \frac{4}{3})$ and $\delta_{m,\Lambda_{\Phi,\omega}}(1) = \frac{2-\gamma}{\gamma}$ for $\gamma \in (\frac{4}{3}, 2)$.

Theorem

Let $0 < \gamma < \infty$ and the numbers γ_0, $u(\Phi,\omega)$ and $v(\Phi,\omega)$ be defined as in Theorem 6. Then the following statements hold true:

(i) *If $\Phi \in \Delta_2(\infty)$, $\gamma_0 = \gamma$ and the weight function ω is regular, then*

$$\varepsilon_{0,m}\left(\Lambda_{\Phi,\omega}\right) = \frac{a(\Phi)}{u(\Phi,\omega)}.$$

(ii) *If $\Phi \in \Delta_2(\infty)$, $\gamma_0 \in (\frac{1}{2}\gamma, \gamma)$ and the weight function ω is regular, then*

$$\varepsilon_{0,m}\left(\Lambda_{\Phi,\omega}\right) = \max\left(\frac{a(\Phi)}{u(\Phi,\omega)}, \frac{u(\Phi,\omega)}{v(\Phi,\omega)}\right).$$

(iii) *If $\Phi \notin \Delta_2(\infty)$ or the weight function ω is not regular, then $\varepsilon_{0,m}\left(\Lambda_{\Phi,\omega}\right) = 1$.*

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Thank you for your attention