On diagonals of commutators of positive compact operators and ideal-triangularizability

Joint work with Roman Drnovšek

Marko Kandić

Faculty of Mathematics and Physics, Ljubljana, Slovenia

Leiden, Netherlands, 22. 7. 2013
Definition

A matrix $A$ is said to be **decomposable** if there exists a permutation matrix $P$ such that $P^TAP$ has the block matrix form

$$
\begin{bmatrix}
* & * \\
0 & *
\end{bmatrix}.
$$

It should be noted that the diagonal entries of the matrix $P^TAP$ are just the permuted diagonal entries of the matrix $A$.  

Definition

A matrix $A$ is said to be *decomposable* if there exists a permutation matrix $P$ such that $P^T AP$ has the block matrix form

$$
\begin{bmatrix}
* & * \\
0 & *
\end{bmatrix}.
$$

It should be noted that the diagonal entries of the matrix $P^T AP$ are just the permuted diagonal entries of the matrix $A$. 
Definition
A matrix $A$ is said to be completely decomposable if there exists a permutation matrix $P$ such that $P^TAP$ is upper-triangular.

- This means that $A$ is upper-triangular upon a permutation similarity.
- By the well known Schur’s theorem from Linear algebra, every complex matrix is similar to an upper-triangular matrix.
Definition

A matrix $A$ is said to be completely decomposable if there exists a permutation matrix $P$ such that $P^T A P$ is upper-triangular.

- This means that $A$ is upper-triangular upon a permutation similarity.
- By the well known Schur’s theorem from Linear algebra every complex matrix is similar to an upper-triangular matrix.
Definition

A matrix $A$ is said to be completely decomposable if there exists a permutation matrix $P$ such that $P^T A P$ is upper-triangular.

- This means that $A$ is upper-triangular upon a permutation similarity.
- By the well known Schur’s theorem from Linear algebra every complex matrix is similar to an upper-triangular matrix.
Eigenvalues of an upper-triangular matrix appear on its diagonal repeated according to their multiplicities.

We conclude that the eigenvalues of a completely decomposable matrix appear on its diagonal repeated according to their multiplicities.

Is the converse statement true?

**Theorem**

Let $A$ be a positive matrix. If the eigenvalues of $A$ appear on the diagonal of $A$ according to their multiplicities, then $A$ is completely decomposable.
Eigenvalues of an upper-triangular matrix appear on its diagonal repeated according to their multiplicities.

We conclude that the eigenvalues of a completely decomposable matrix appear on its diagonal repeated according to their multiplicities.

Is the converse statement true?

**Theorem**

Let $A$ be a positive matrix. If the eigenvalues of $A$ appear on the diagonal of $A$ according to their multiplicities, then $A$ is completely decomposable.
Eigenvalues of an upper-triangular matrix appear on its diagonal repeated according to their multiplicities.

We conclude that the eigenvalues of a completely decomposable matrix appear on its diagonal repeated according to their multiplicities.

Is the converse statement true?

**Theorem**

Let $A$ be a positive matrix. If the eigenvalues of $A$ appear on the diagonal of $A$ according to their multiplicities, then $A$ is completely decomposable.
Eigenvalues of an upper-triangular matrix appear on its diagonal repeated according to their multiplicities.

We conclude that the eigenvalues of a completely decomposable matrix appear on its diagonal repeated according to their multiplicities.

Is the converse statement true?

**Theorem**

Let $A$ be a positive matrix. If the eigenvalues of $A$ appear on the diagonal of $A$ according to their multiplicities, then $A$ is completely decomposable.
Let $A$ and $B$ be simultaneously completely decomposable matrices.

- With respect to some permutation of the standard basis of the underlying space, the matrix $AB - BA$ is strictly upper-triangular.
- The diagonal of the matrix $AB - BA$ is zero.

We will see later that the converse statement holds in some cases.
Let $A$ and $B$ be simultaneously completely decomposable matrices.

- With respect to some permutation of the standard basis of the underlying space, the matrix $AB - BA$ is strictly upper-triangular.

- The diagonal of the matrix $AB - BA$ is zero.

We will see later that the converse statement holds in some cases.
Let $A$ and $B$ be simultaneously completely decomposable matrices.

- With respect to some permutation of the standard basis of the underlying space, the matrix $AB - BA$ is strictly upper-triangular.
- The diagonal of the matrix $AB - BA$ is zero.

We will see later that the converse statement holds in some cases.
Let $A$ and $B$ be simultaneously completely decomposable matrices.

- With respect to some permutation of the standard basis of the underlying space, the matrix $AB - BA$ is strictly upper-triangular.
- The diagonal of the matrix $AB - BA$ is zero.

We will see later that the converse statement holds in some cases.
Let $E$ be a normed Riesz space and $\mathcal{F}$ a family of bounded operators on $E$.

- $\mathcal{F}$ is ideal-reducible if there exists a closed ideal in $E$ that is invariant under every operator from $\mathcal{F}$.
- $\mathcal{F}$ is ideal-triangularizable if there exists a maximal chain $\mathcal{C}$ of closed ideals in $E$ such that every ideal from $\mathcal{C}$ is invariant under every operator from $\mathcal{F}$.

**Theorem (Drnovšek 2000)**

*Every maximal chain of closed ideals in $E$ is also a maximal chain of closed subspaces in $E$.*
Let $E$ be a normed Riesz space and $\mathcal{F}$ a family of bounded operators on $E$.

- $\mathcal{F}$ is ideal-reducible if there exists a closed ideal in $E$ that is invariant under every operator from $\mathcal{F}$.

- $\mathcal{F}$ is ideal-triangularizable if there exists a maximal chain $\mathcal{C}$ of closed ideals in $E$ such that every ideal from $\mathcal{C}$ is invariant under every operator from $\mathcal{F}$.

**Theorem (Drnovšek 2000)**

*Every maximal chain of closed ideals in $E$ is also a maximal chain of closed subspaces in $E$.*
Let $E$ be a normed Riesz space and $\mathcal{F}$ a family of bounded operators on $E$.

- $\mathcal{F}$ is ideal-reducible if there exists a closed ideal in $E$ that is invariant under every operator from $\mathcal{F}$.
- $\mathcal{F}$ is ideal-triangularizable if there exists a maximal chain $\mathcal{C}$ of closed ideals in $E$ such that every ideal from $\mathcal{C}$ is invariant under every operator from $\mathcal{F}$.

**Theorem (Drnovšek 2000)**

*Every maximal chain of closed ideals in $E$ is also a maximal chain of closed subspaces in $E$.*
Let $E$ be a normed Riesz space and $\mathcal{F}$ a family of bounded operators on $E$.

- $\mathcal{F}$ is ideal-reducible if there exists a closed ideal in $E$ that is invariant under every operator from $\mathcal{F}$.
- $\mathcal{F}$ is ideal-triangularizable if there exists a maximal chain $\mathcal{C}$ of closed ideals in $E$ such that every ideal from $\mathcal{C}$ is invariant under every operator from $\mathcal{F}$.

**Theorem (Drnovšek 2000)**

*Every maximal chain of closed ideals in $E$ is also a maximal chain of closed subspaces in $E$.*
Example

Let $n \in \mathbb{N}$ be arbitrary. Consider the Banach space $(\mathbb{R}^n, \| \cdot \|_\infty)$.

- If we define ordering on $\mathbb{R}^n$ componentwise, then $(\mathbb{R}^n, \| \cdot \|_\infty)$ is a Banach lattice.
- Closed ideals in $\mathbb{R}^n$ are precisely those of the form
  \[ \text{lin}\{ e_j : j \in J \subseteq \{1, \ldots, n\} \} \]

- Ideal-triangularizability is in this case the same as complete decomposability.
Example

Let $n \in \mathbb{N}$ be arbitrary. Consider the Banach space $(\mathbb{R}^n, \| \cdot \|_\infty)$. If we define ordering on $\mathbb{R}^n$ componentwise, then $(\mathbb{R}^n, \| \cdot \|_\infty)$ is a Banach lattice.

- Closed ideals in $\mathbb{R}^n$ are precisely those of the form

$$\text{lin}\{ e_j : j \in J \subseteq \{1, \ldots, n\} \}.$$

- Ideal-triangularizability is in this case the same as complete decomposability.
Example

Let $n \in \mathbb{N}$ be arbitrary. Consider the Banach space $(\mathbb{R}^n, \| \cdot \|_\infty)$.

- If we define ordering on $\mathbb{R}^n$ componentwise, then $(\mathbb{R}^n, \| \cdot \|_\infty)$ is a Banach lattice.
- Closed ideals in $\mathbb{R}^n$ are precisely those of the form $\operatorname{lin}\{ e_j : j \in J \subseteq \{1, \ldots, n\} \}$.

- Ideal-triangularizability is in this case the same as complete decomposability.
Example

Let $n \in \mathbb{N}$ be arbitrary. Consider the Banach space $(\mathbb{R}^n, \| \cdot \|_\infty)$.

- If we define ordering on $\mathbb{R}^n$ componentwise, then $(\mathbb{R}^n, \| \cdot \|_\infty)$ is a Banach lattice.
- Closed ideals in $\mathbb{R}^n$ are precisely those of the form

$$\text{lin}\{e_j : j \in J \subseteq \{1, \ldots, n\}\}.$$

- Ideal-triangularizability is in this case the same as complete decomposability.
A subset \( I \) of a semigroup \( S \) is said to be a **semigroup ideal** if \( ST \) and \( TS \) belong to \( I \) for all \( S \in S \) and \( T \in I \).

**Proposition (Drnovšek, Kandić 2009)**

Let \( E \) be a normed Riesz space, and let \( S \) be a nonzero semigroup of positive operators on \( E \). The following statements are equivalent:

1. \( S \) is ideal-reducible;
2. there exist a nonzero positive functional \( \varphi \in E^* \) and a nonzero positive vector \( f \in E \) such that \( \varphi(Sf) = \{0\} \);
3. there exist nonzero positive operators \( A \) and \( B \) on \( E \) such that \( A S B = \{0\} \);
4. some nonzero semigroup ideal of \( S \) is ideal-reducible.
A subset $\mathcal{I}$ of a semigroup $\mathcal{S}$ is said to be a semigroup ideal if $ST$ and $TS$ belong to $\mathcal{I}$ for all $S \in \mathcal{S}$ and $T \in \mathcal{I}$.

**Proposition (Drnovšek, Kandič 2009)**

Let $E$ be a normed Riesz space, and let $\mathcal{S}$ be a nonzero semigroup of positive operators on $E$. The following statements are equivalent:

1. $\mathcal{S}$ is ideal-reducible;
2. there exist a nonzero positive functional $\varphi \in E^*$ and a nonzero positive vector $f \in E$ such that $\varphi(\mathcal{S}f) = \{0\}$;
3. there exist nonzero positive operators $A$ and $B$ on $E$ such that $A\mathcal{S}B = \{0\}$;
4. some nonzero semigroup ideal of $\mathcal{S}$ is ideal-reducible.
A subset $\mathcal{I}$ of a semigroup $\mathcal{I}$ is said to be a *semigroup ideal* if $ST$ and $TS$ belong to $\mathcal{I}$ for all $S \in \mathcal{I}$ and $T \in \mathcal{I}$.

**Proposition (Drnovšek, Kandić 2009)**

Let $E$ be a normed Riesz space, and let $\mathcal{I}$ be a nonzero semigroup of positive operators on $E$. The following statements are equivalent:

1. $\mathcal{I}$ is ideal-reducible;
2. there exist a nonzero positive functional $\varphi \in E^*$ and a nonzero positive vector $f \in E$ such that $\varphi(\mathcal{I} f) = \{0\}$;
3. there exist nonzero positive operators $A$ and $B$ on $E$ such that $A\mathcal{I}B = \{0\}$;
4. some nonzero semigroup ideal of $\mathcal{I}$ is ideal-reducible.
A subset $\mathcal{I}$ of a semigroup $\mathcal{S}$ is said to be a semigroup ideal if $ST$ and $TS$ belong to $\mathcal{I}$ for all $S \in \mathcal{S}$ and $T \in \mathcal{I}$.

**Proposition (Drnovšek, Kandić 2009)**

Let $E$ be a normed Riesz space, and let $\mathcal{S}$ be a nonzero semigroup of positive operators on $E$. The following statements are equivalent:

1. $\mathcal{S}$ is ideal-reducible;
2. there exist a nonzero positive functional $\varphi \in E^*$ and a nonzero positive vector $f \in E$ such that $\varphi(\mathcal{S}f) = \{0\}$;
3. there exist nonzero positive operators $A$ and $B$ on $E$ such that $A\mathcal{S}B = \{0\}$;
4. some nonzero semigroup ideal of $\mathcal{S}$ is ideal-reducible.
A subset $\mathcal{I}$ of a semigroup $\mathcal{S}$ is said to be a *semigroup ideal* if $ST$ and $TS$ belong to $\mathcal{I}$ for all $S \in \mathcal{S}$ and $T \in \mathcal{I}$.

**Proposition (Drnovšek, Kandić 2009)**

Let $E$ be a normed Riesz space, and let $\mathcal{S}$ be a nonzero semigroup of positive operators on $E$. The following statements are equivalent:

1. $\mathcal{S}$ is ideal-reducible;
2. there exist a nonzero positive functional $\varphi \in E^*$ and a nonzero positive vector $f \in E$ such that $\varphi(\mathcal{S}f) = \{0\}$;
3. there exist nonzero positive operators $A$ and $B$ on $E$ such that $A\mathcal{S}B = \{0\}$;
4. some nonzero semigroup ideal of $\mathcal{S}$ is ideal-reducible.
A subset $\mathcal{I}$ of a semigroup $\mathcal{I}$ is said to be a \textit{semigroup ideal} if $ST$ and $TS$ belong to $\mathcal{I}$ for all $S \in \mathcal{I}$ and $T \in \mathcal{I}$.

**Proposition (Drnovšek, Kandić 2009)**

Let $E$ be a normed Riesz space, and let $\mathcal{I}$ be a nonzero semigroup of positive operators on $E$. The following statements are equivalent:

1. $\mathcal{I}$ is ideal-reducible;
2. there exist a nonzero positive functional $\varphi \in E^*$ and a nonzero positive vector $f \in E$ such that $\varphi(\mathcal{I}f) = \{0\}$;
3. there exist nonzero positive operators $A$ and $B$ on $E$ such that $A\mathcal{I}B = \{0\}$;
4. some nonzero semigroup ideal of $\mathcal{I}$ is ideal-reducible.
Let \( \mathcal{F} \) be a family of bounded linear operators on a Banach lattice \( E \), and \( \mathcal{I}, \mathcal{J} \) closed ideals invariant under \( \mathcal{F} \) with \( \mathcal{I} \subseteq \mathcal{J} \). Then the operator \( \hat{T} : \mathcal{J}/\mathcal{I} \to \mathcal{J}/\mathcal{I} \) defined by 
\[
\hat{T}(x + \mathcal{I}) = Tx + \mathcal{I}
\]
is well defined for each \( x \in \mathcal{J} \).

**Definition**

A property of families of operators on a Banach lattice is said to be inherited by ideal-quotients if for each family \( \mathcal{F} \) having the property, and for every distinct pair \( \mathcal{I}, \mathcal{J} \) of closed ideals invariant under \( \mathcal{F} \) with \( \mathcal{I} \subseteq \mathcal{J} \), the family \( \hat{\mathcal{F}} \) also has the property, where \( \hat{\mathcal{F}} \) is the set of all quotient operators \( \hat{T} \) on \( \mathcal{J}/\mathcal{I} \) for \( T \in \mathcal{F} \).
Let $\mathcal{F}$ be a family of bounded linear operators on a Banach lattice $E$, and $\mathcal{I}, \mathcal{J}$ closed ideals invariant under $\mathcal{F}$ with $\mathcal{I} \subseteq \mathcal{J}$. Then the operator $\hat{T} : \mathcal{J}/\mathcal{I} \to \mathcal{J}/\mathcal{I}$ defined by $\hat{T}(x + \mathcal{I}) = Tx + \mathcal{I}$ is well defined for each $x \in \mathcal{J}$.

**Definition**

A property of families of operators on a Banach lattice is said to be inherited by ideal-quotients if for each family $\mathcal{F}$ having the property, and for every distinct pair $\mathcal{I}, \mathcal{J}$ of closed ideals invariant under $\mathcal{F}$ with $\mathcal{I} \subseteq \mathcal{J}$, the family $\hat{\mathcal{F}}$ also has the property, where $\hat{\mathcal{F}}$ is the set of all quotient operators $\hat{T}$ on $\mathcal{J}/\mathcal{I}$ for $T \in \mathcal{F}$. 

The following lemma is the key to obtain ideal-triangularizing chains for families of operators.

**Lemma (The Ideal-Triangularization Lemma)**

If $\mathcal{P}$ is a property of families of operators that is inherited by ideal-quotients, and if every family satisfying $\mathcal{P}$ on a Banach lattice of dimension at least two is ideal-reducible, then every family satisfying $\mathcal{P}$ is ideal-triangularizable.
The following lemma is the key to obtain ideal-triangularizing chains for families of operators.

**Lemma (The Ideal-Triangularization Lemma)**

*If $\mathcal{P}$ is a property of families of operators that is inherited by ideal-quotients, and if every family satisfying $\mathcal{P}$ on a Banach lattice of dimension at least two is ideal-reducible, then every family satisfying $\mathcal{P}$ is ideal-triangularizable.*
Which properties are inherited by ideal-quotients?

- commutativity
- nilpotence
- quasinilpotence
- compactness
- weak compactness
- positivity
- being a Riesz operator
- being an abstract integral operator on a Banach lattice with order continuous norm
Which properties are inherited by ideal-quotients?

- commutativity
- nilpotence
- quasinilpotence
- compactness
- weak compactness
- positivity
- being a Riesz operator
- being an abstract integral operator on a Banach lattice with order continuous norm
Which properties are inherited by ideal-quotients?

- commutativity
- nilpotence
- quasinilpotence
- compactness
- weak compactness
- positivity
- being a Riesz operator
- being an abstract integral operator on a Banach lattice with order continuous norm
Which properties are inherited by ideal-quotients?

- commutativity
- nilpotence
- quasinilpotence
- compactness
- weak compactness
- positivity
- being a Riesz operator
- being an abstract integral operator on a Banach lattice with order continuous norm
Which properties are inherited by ideal-quotients?

- commutativity
- nilpotence
- quasinilpotence
- compactness
- weak compactness
- positivity
- being a Riesz operator
- being an abstract integral operator on a Banach lattice with order continuous norm
Which properties are inherited by ideal-quotients?

- commutativity
- nilpotence
- quasinilpotence
- compactness
- weak compactness
- positivity
- being a Riesz operator
- being an abstract integral operator on a Banach lattice with order continuous norm
Which properties are inherited by ideal-quotients?

- commutativity
- nilpotence
- quasinilpotence
- compactness
- weak compactness
- positivity
- being a Riesz operator
- being an abstract integral operator on a Banach lattice with order continuous norm
Which properties are inherited by ideal-quotients?

- commutativity
- nilpotence
- quasinilpotence
- compactness
- weak compactness
- positivity
- being a Riesz operator
- being an abstract integral operator on a Banach lattice with order continuous norm
Which properties are inherited by ideal-quotients?

- commutativity
- nilpotence
- quasinilpotence
- compactness
- weak compactness
- positivity
- being a Riesz operator
- being an abstract integral operator on a Banach lattice with order continuous norm
Let us see an example how in practice the Ideal-triangularization lemma works.

Every positive nilpotent operator $T$ on a Banach lattice $E$ is ideal-triangularizable.

- If $T$ is a zero operator, then every closed ideal is invariant under $T$.
- Assume $T$ is nonzero.
- The absolute kernel $N(T) = \{ x \in E : T|x| = 0 \}$ is a nonzero closed ideal invariant under $T$, so that $T$ is ideal-reducible.
- Nilpotency of operators is inherited by ideal-quotients.
- Apply the Ideal-triangularization lemma.
Let us see an example how in practice the Ideal-triangularization lemma works.

Every positive nilpotent operator $T$ on a Banach lattice $E$ is ideal-triangularizable.

- If $T$ is a zero operator, then every closed ideal is invariant under $T$.
- Assume $T$ is nonzero.
- The absolute kernel $N(T) = \{ x \in E : T|x| = 0 \}$ is a nonzero closed ideal invariant under $T$, so that $T$ is ideal-reducible.
- Nilpotency of operators is inherited by ideal-quotients.
- Apply the Ideal-triangularization lemma.
Let us see an example how in practice the Ideal-triangularization lemma works.

Every positive nilpotent operator $T$ on a Banach lattice $E$ is ideal-triangularizable.

- If $T$ is a zero operator, then every closed ideal is invariant under $T$.
- Assume $T$ is nonzero.
- The absolute kernel $N(T) = \{ x \in E : T|x| = 0 \}$ is a nonzero closed ideal invariant under $T$, so that $T$ is ideal-reducible.
- Nilpotency of operators is inherited by ideal-quotients.
- Apply the Ideal-triangularization lemma.
Let us see an example how in practice the Ideal-triangularization lemma works.

Every positive nilpotent operator $T$ on a Banach lattice $E$ is ideal-triangularizable.

- If $T$ is a zero operator, then every closed ideal is invariant under $T$.
- Assume $T$ is nonzero.
  - The absolute kernel $N(T) = \{ x \in E : T|x| = 0 \}$ is a nonzero closed ideal invariant under $T$, so that $T$ is ideal-reducible.
  - Nilpotency of operators is inherited by ideal-quotients.
  - Apply the Ideal-triangularization lemma.
Let us see an example how in practice the Ideal-triangularization lemma works.

Every positive nilpotent operator $T$ on a Banach lattice $E$ is ideal-triangularizable.

- If $T$ is a zero operator, then every closed ideal is invariant under $T$.
- Assume $T$ is nonzero.
- The absolute kernel $N(T) = \{ x \in E : T|x| = 0 \}$ is a nonzero closed ideal invariant under $T$, so that $T$ is ideal-reducible.
- Nilpotency of operators is inherited by ideal-quotients.
- Apply the Ideal-triangularization lemma.
Let us see an example how in practice the Ideal-triangularization lemma works.

Every positive nilpotent operator $T$ on a Banach lattice $E$ is ideal-triangularizable.

- If $T$ is a zero operator, then every closed ideal is invariant under $T$.
- Assume $T$ is nonzero.
- The absolute kernel $N(T) = \{ x \in E : T|x| = 0 \}$ is a nonzero closed ideal invariant under $T$, so that $T$ is ideal-reducible.
- Nilpotency of operators is inherited by ideal-quotients.
- Apply the Ideal-triangularization lemma.
Let us see an example how in practice the Ideal-triangularization lemma works.

Every positive nilpotent operator $T$ on a Banach lattice $E$ is ideal-triangularizable.

- If $T$ is a zero operator, then every closed ideal is invariant under $T$.
- Assume $T$ is nonzero.
- The absolute kernel $N(T) = \{ x \in E : T|x| = 0 \}$ is a nonzero closed ideal invariant under $T$, so that $T$ is ideal-reducible.
- Nilpotency of operators is inherited by ideal-quotients.
- Apply the Ideal-triangularization lemma.
Let $M_n(\mathbb{R})$ be a vector space of all real $n \times n$ matrices. The mapping $\varphi_j$ that maps every matrix $S$ to the $j$-th diagonal entry of $S$ is a positive linear functional on $M_n(\mathbb{R})$. Suppose that $\mathcal{S}$ is a multiplicative semigroup of positive matrices in $M_n(\mathbb{R})$. If there exists a positive number $j$ with $1 \leq j \leq n$ such that $\varphi_j(S) = 0$ for all $S \in \mathcal{S}$, then $\mathcal{S}$ is ideal-reducible. This is a direct consequence of the characterization of ideal-reducibility of semigroups of positive operators, since we have

$$\varphi_j(S) = \langle Se_j, e_j \rangle = e_j^T Se_j.$$ 

Actually, we can say even more about ideal-reducibility and ideal-triangularizability of semigroups of positive matrices.
Recall that a map \( \varphi : \mathcal{S}_1 \to \mathcal{S}_2 \) between two semigroups \( \mathcal{S}_1 \) and \( \mathcal{S}_2 \) is multiplicative whenever \( \varphi(ab) = \varphi(a)\varphi(b) \) for all \( a, b \in \mathcal{S}_1 \).

**Theorem**

Let \( \mathcal{S} \) be a semigroup of positive \( n \times n \) matrices. Then the following statements hold.

- If \( \varphi_j \) is multiplicative on \( \mathcal{S} \) for some \( 1 \leq j \leq n \), then \( \mathcal{S} \) is ideal-reducible.
- \( \mathcal{S} \) is ideal-triangularizable if and only if \( \varphi_j \) is multiplicative on \( \mathcal{S} \) for \( 1 \leq j \leq n \).
Recall that a map $\varphi : \mathcal{S}_1 \rightarrow \mathcal{S}_2$ between two semigroups $\mathcal{S}_1$ and $\mathcal{S}_2$ is multiplicative whenever $\varphi(ab) = \varphi(a)\varphi(b)$ for all $a, b \in \mathcal{S}_1$.

**Theorem**

Let $\mathcal{I}$ be a semigroup of positive $n \times n$ matrices. Then the following statements hold.

- If $\varphi_j$ is multiplicative on $\mathcal{I}$ for some $1 \leq j \leq n$, then $\mathcal{I}$ is ideal-reducible.
- $\mathcal{I}$ is ideal-triangularizable if and only if $\varphi_j$ is multiplicative on $\mathcal{I}$ for $1 \leq j \leq n$. 
Recall that a map $\varphi : S_1 \rightarrow S_2$ between two semigroups $S_1$ and $S_2$ is multiplicative whenever $\varphi(ab) = \varphi(a)\varphi(b)$ for all $a, b \in S_1$.

**Theorem**

Let $\mathcal{S}$ be a semigroup of positive $n \times n$ matrices. Then the following statements hold.

- If $\varphi_j$ is multiplicative on $\mathcal{S}$ for some $1 \leq j \leq n$, then $\mathcal{S}$ is ideal-reducible.
- $\mathcal{S}$ is ideal-triangularizable if and only if $\varphi_j$ is multiplicative on $\mathcal{S}$ for $1 \leq j \leq n$. 
Recall that a map $\varphi : S_1 \to S_2$ between two semigroups $S_1$ and $S_2$ is multiplicative whenever $\varphi(ab) = \varphi(a)\varphi(b)$ for all $a, b \in S_1$.

**Theorem**

Let $\mathcal{S}$ be a semigroup of positive $n \times n$ matrices. Then the following statements hold.

- If $\varphi_j$ is multiplicative on $\mathcal{S}$ for some $1 \leq j \leq n$, then $\mathcal{S}$ is ideal-reducible.
- $\mathcal{S}$ is ideal-triangularizable if and only if $\varphi_j$ is multiplicative on $\mathcal{S}$ for $1 \leq j \leq n$. 
Let $E$ be a Riesz space.

- A positive vector $a \in E$ is said to be an atom if $0 \leq x, y \leq a$ and $x \wedge y = 0$ imply $x = 0$ or $y = 0$.
- In Archimedean spaces atoms are precisely discrete vectors, i.e., a positive vector $a \in E$ is said to be a discrete vector if $0 \leq x \leq a$ imply the existence of a nonnegative scalar $\lambda$ such that $x = \lambda a$.
- The latter implies that the principal ideal $\mathcal{B}_a$ generated by $a$ is one dimensional projection band.
- $E$ is said to be atomic if the band $A$ generated by all atoms of $E$ is equal to $E$. If $A = \{0\}$, then $E$ is atomless.
- If $E$ is Dedekind complete, then $C := A^d$ is an atomless part of the lattice $E$. 
Let $E$ be a Riesz space.

- A positive vector $a \in E$ is said to be an atom if $0 \leq x, y \leq a$ and $x \wedge y = 0$ imply $x = 0$ or $y = 0$.

- In Archimedean spaces atoms are precisely discrete vectors, i.e., a positive vector $a \in E$ is said to be a discrete vector if $0 \leq x \leq a$ imply the existence of a nonnegative scalar $\lambda$ such that $x = \lambda a$.

- The latter implies that the principal ideal $\mathcal{B}_a$ generated by $a$ is one dimensional projection band.

- $E$ is said to be atomic if the band $A$ generated by all atoms of $E$ is equal to $E$. If $A = \{0\}$, then $E$ is atomless.

- If $E$ is Dedekind complete, then $C := A^d$ is an atomless part of the lattice $E$. 
Let $E$ be a Riesz space.

- A positive vector $a \in E$ is said to be an atom if $0 \leq x, y \leq a$ and $x \wedge y = 0$ imply $x = 0$ or $y = 0$.
- In Archimedean spaces atoms are precisely discrete vectors, i.e., a positive vector $a \in E$ is said to be a discrete vector if $0 \leq x \leq a$ imply the existence of a nonnegative scalar $\lambda$ such that $x = \lambda a$.

- The latter implies that the principal ideal $B_a$ generated by $a$ is one dimensional projection band.
- $E$ is said to be atomic if the band $A$ generated by all atoms of $E$ is equal to $E$. If $A = \{0\}$, then $E$ is atomless.
- If $E$ is Dedekind complete, then $C := A^d$ is an atomless part of the lattice $E$. 
Let $E$ be a Riesz space.

- A positive vector $a \in E$ is said to be an atom if $0 \leq x, y \leq a$ and $x \land y = 0$ imply $x = 0$ or $y = 0$.

- In Archimedean spaces atoms are precisely discrete vectors, i.e., a positive vector $a \in E$ is said to be a discrete vector if $0 \leq x \leq a$ imply the existence of a nonnegative scalar $\lambda$ such that $x = \lambda a$.

- The latter implies that the principal ideal $\mathcal{B}_a$ generated by $a$ is one dimensional projection band.

- $E$ is said to be atomic if the band $A$ generated by all atoms of $E$ is equal to $E$. If $A = \{0\}$, then $E$ is atomless.

- If $E$ is Dedekind complete, then $C := A^d$ is an atomless part of the lattice $E$. 


Let $E$ be a Riesz space.

- A positive vector $a \in E$ is said to be an atom if $0 \leq x, y \leq a$ and $x \wedge y = 0$ imply $x = 0$ or $y = 0$.
- In Archimedean spaces atoms are precisely discrete vectors, i.e., a positive vector $a \in E$ is said to be a discrete vector if $0 \leq x \leq a$ imply the existence of a nonnegative scalar $\lambda$ such that $x = \lambda a$.
- The latter implies that the principal ideal $\mathcal{B}_a$ generated by $a$ is one dimensional projection band.
- $E$ is said to be atomic if the band $A$ generated by all atoms of $E$ is equal to $E$. If $A = \{0\}$, then $E$ is atomless.
- If $E$ is Dedekind complete, then $C := A^d$ is an atomless part of the lattice $E$. 
Let $E$ be a Riesz space.

- A positive vector $a \in E$ is said to be an atom if $0 \leq x, y \leq a$ and $x \wedge y = 0$ imply $x = 0$ or $y = 0$.
- In Archimedean spaces atoms are precisely discrete vectors, i.e., a positive vector $a \in E$ is said to be a discrete vector if $0 \leq x \leq a$ imply the existence of a nonnegative scalar $\lambda$ such that $x = \lambda a$.
- The latter implies that the principal ideal $B_a$ generated by $a$ is one dimensional projection band.
- $E$ is said to be atomic if the band $A$ generated by all atoms of $E$ is equal to $E$. If $A = \{0\}$, then $E$ is atomless.
- If $E$ is Dedekind complete, then $C := A^d$ is an atomless part of the lattice $E$. 
Suppose $E$ is a normed Riesz space and $a$ an atom in $E$. Then $E = \mathcal{B}_a \oplus \mathcal{B}_a^d$ is an order direct sum of two bands of $E$.

- For a (positive) $x \in E$ there exist a (positive) scalar $\lambda_x$ and a (positive) vector $y_x \in \mathcal{B}_a^d$ such that $x = \lambda_x a + y_x$.
- The mapping $\varphi_a : E \to \mathbb{R}$ defined by $\varphi_a(x) = \lambda_x$ is a bounded positive linear functional on $E$.
- $\varphi_a(STa) \geq \varphi_a(Sa) \varphi_a(Ta)$ for positive operators $S$ and $T$ on $E$.
- If $E = \mathbb{R}^n$, then the normalized atoms of $E$ are precisely the standard basis vectors. If $S = [s_{ij}]_{i,j=1}^n$ is a positive matrix and $a$ is an atom in $\mathbb{R}^n$, then $\varphi_a(Sa) = s_{jj}$ for some $1 \leq j \leq n$. 
Suppose $E$ is a normed Riesz space and $a$ an atom in $E$. Then $E = \mathcal{B}_a \oplus \mathcal{B}^d_a$ is an order direct sum of two bands of $E$.

- For a (positive) $x \in E$ there exist a (positive) scalar $\lambda_x$ and a (positive) vector $y_x \in \mathcal{B}^d_a$ such that $x = \lambda_x a + y_x$.

- The mapping $\varphi_a : E \rightarrow \mathbb{R}$ defined by $\varphi_a(x) = \lambda_x$ is a bounded positive linear functional on $E$.

- $\varphi_a(STa) \geq \varphi_a(Sa)\varphi_a(Ta)$ for positive operators $S$ and $T$ on $E$.

- If $E = \mathbb{R}^n$, then the normalized atoms of $E$ are precisely the standard basis vectors. If $S = [s_{ij}]_{i,j=1}^n$ is a positive matrix and $a$ is an atom in $\mathbb{R}^n$, then $\varphi_a(Sa) = s_{jj}$ for some $1 \leq j \leq n$. 
Suppose $E$ is a normed Riesz space and $a$ an atom in $E$. Then $E = \mathcal{B}_a \oplus \mathcal{B}_a^d$ is an order direct sum of two bands of $E$.

- For a (positive) $x \in E$ there exist a (positive) scalar $\lambda_x$ and a (positive) vector $y_x \in \mathcal{B}_a^d$ such that $x = \lambda_x a + y_x$.
- The mapping $\varphi_a : E \to \mathbb{R}$ defined by $\varphi_a(x) = \lambda_x$ is a bounded positive linear functional on $E$.
- $\varphi_a(STa) \geq \varphi_a(Sa)\varphi_a(Ta)$ for positive operators $S$ and $T$ on $E$.
- If $E = \mathbb{R}^n$, then the normalized atoms of $E$ are precisely the standard basis vectors. If $S = [s_{ij}]_{i,j=1}^n$ is a positive matrix and $a$ is an atom in $\mathbb{R}^n$, then $\varphi_a(Sa) = s_{jj}$ for some $1 \leq j \leq n$. 
Suppose $E$ is a normed Riesz space and $a$ an atom in $E$. Then $E = B_a \oplus B_a^d$ is an order direct sum of two bands of $E$.

- For a (positive) $x \in E$ there exist a (positive) scalar $\lambda_x$ and a (positive) vector $y_x \in B_a^d$ such that $x = \lambda_x a + y_x$.
- The mapping $\varphi_a : E \to \mathbb{R}$ defined by $\varphi_a(x) = \lambda_x$ is a bounded positive linear functional on $E$.
- $\varphi_a(STa) \geq \varphi_a(Sa)\varphi_a(Ta)$ for positive operators $S$ and $T$ on $E$.
- If $E = \mathbb{R}^n$, then the normalized atoms of $E$ are precisely the standard basis vectors. If $S = [s_{ij}]_{i,j=1}^n$ is a positive matrix and $a$ is an atom in $\mathbb{R}^n$, then $\varphi_a(Sa) = s_{jj}$ for some $1 \leq j \leq n$. 

Suppose $E$ is a normed Riesz space and $a$ an atom in $E$. Then $E = \mathcal{B}_a \oplus \mathcal{B}_a^d$ is an order direct sum of two bands of $E$.

- For a (positive) $x \in E$ there exist a (positive) scalar $\lambda_x$ and a (positive) vector $y_x \in \mathcal{B}_a^d$ such that $x = \lambda_x a + y_x$.

- The mapping $\varphi_a : E \to \mathbb{R}$ defined by $\varphi_a(x) = \lambda_x$ is a bounded positive linear functional on $E$.

- $\varphi_a(STa) \geq \varphi_a(Sa) \varphi_a(Ta)$ for positive operators $S$ and $T$ on $E$.

- If $E = \mathbb{R}^n$, then the normalized atoms of $E$ are precisely the standard basis vectors. If $S = [s_{ij}]_{i,j=1}^n$ is a positive matrix and $a$ is an atom in $\mathbb{R}^n$, then $\varphi_a(Sa) = s_{jj}$ for some $1 \leq j \leq n$. 
Proposition

Let $E$ be a normed Riesz space and $\mathcal{I}$ a semigroup of positive operators on $E$.

1. If $\mathcal{I}$ is ideal-triangularizable, then the mapping $S \mapsto \varphi_a(Sa)$ is multiplicative on $\mathcal{I}$ for every atom $a \in E$.

2. If the mapping $S \mapsto \varphi_a(Sa)$ is multiplicative on $\mathcal{I}$ for some atom $a \in E$, then $\mathcal{I}$ is ideal-reducible.

3. If $E$ is atomic Banach lattice with order continuous norm, then $\mathcal{I}$ is ideal-triangularizable if and only if the mapping $S \mapsto \varphi_a(Sa)$ is multiplicative on $\mathcal{I}$ for all atoms $a \in E$.

The mapping $S \mapsto \varphi_a(Sa)$ is called a coordinate functional associated to an atom $a$, and the number $\varphi_a(Sa)$ is called the diagonal entry of $S$ associated to the atom $a$. 
Proposition

Let $E$ be a normed Riesz space and $\mathcal{S}$ a semigroup of positive operators on $E$.

1. If $\mathcal{S}$ is ideal-triangularizable, then the mapping $S \mapsto \varphi_a(Sa)$ is multiplicative on $\mathcal{S}$ for every atom $a \in E$.

2. If the mapping $S \mapsto \varphi_a(Sa)$ is multiplicative on $\mathcal{S}$ for some atom $a \in E$, then $\mathcal{S}$ is ideal-reducible.

3. If $E$ is atomic Banach lattice with order continuous norm, then $\mathcal{S}$ is ideal-triangularizable if and only if the mapping $S \mapsto \varphi_a(Sa)$ is multiplicative on $\mathcal{S}$ for all atoms $a \in E$.

The mapping $S \mapsto \varphi_a(Sa)$ is called a coordinate functional associated to an atom $a$, and the number $\varphi_a(Sa)$ is called the diagonal entry of $S$ associated to the atom $a$. 
Proposition

Let $E$ be a normed Riesz space and $\mathcal{I}$ a semigroup of positive operators on $E$.

1. If $\mathcal{I}$ is ideal-triangularizable, then the mapping $S \mapsto \varphi_a(Sa)$ is multiplicative on $\mathcal{I}$ for every atom $a \in E$.

2. If the mapping $S \mapsto \varphi_a(Sa)$ is multiplicative on $\mathcal{I}$ for some atom $a \in E$, then $\mathcal{I}$ is ideal-reducible.

3. If $E$ is atomic Banach lattice with order continuous norm, then $\mathcal{I}$ is ideal-triangularizable if and only if the mapping $S \mapsto \varphi_a(Sa)$ is multiplicative on $\mathcal{I}$ for all atoms $a \in E$.

The mapping $S \mapsto \varphi_a(Sa)$ is called a coordinate functional associated to an atom $a$, and the number $\varphi_a(Sa)$ is called the diagonal entry of $S$ associated to the atom $a$. 
Proposition

Let $E$ be a normed Riesz space and $\mathcal{I}$ a semigroup of positive operators on $E$.

1. If $\mathcal{I}$ is ideal-triangularizable, then the mapping $S \mapsto \phi_a(Sa)$ is multiplicative on $\mathcal{I}$ for every atom $a \in E$.

2. If the mapping $S \mapsto \phi_a(Sa)$ is multiplicative on $\mathcal{I}$ for some atom $a \in E$, then $\mathcal{I}$ is ideal-reducible.

3. If $E$ is atomic Banach lattice with order continuous norm, then $\mathcal{I}$ is ideal-triangularizable if and only if the mapping $S \mapsto \phi_a(Sa)$ is multiplicative on $\mathcal{I}$ for all atoms $a \in E$.

The mapping $S \mapsto \phi_a(Sa)$ is called a coordinate functional associated to an atom $a$, and the number $\phi_a(Sa)$ is called the diagonal entry of $S$ associated to the atom $a$. 
Proposition

Let $E$ be a normed Riesz space and $\mathcal{I}$ a semigroup of positive operators on $E$.

1. If $\mathcal{I}$ is ideal-triangularizable, then the mapping $S \mapsto \varphi_a(Sa)$ is multiplicative on $\mathcal{I}$ for every atom $a \in E$.

2. If the mapping $S \mapsto \varphi_a(Sa)$ is multiplicative on $\mathcal{I}$ for some atom $a \in E$, then $\mathcal{I}$ is ideal-reducible.

3. If $E$ is atomic Banach lattice with order continuous norm, then $\mathcal{I}$ is ideal-triangularizable if and only if the mapping $S \mapsto \varphi_a(Sa)$ is multiplicative on $\mathcal{I}$ for all atoms $a \in E$.

The mapping $S \mapsto \varphi_a(Sa)$ is called a coordinate functional associated to an atom $a$, and the number $\varphi_a(Sa)$ is called the diagonal entry of $S$ associated to the atom $a$. 
If the lattice is not atomic (3) does not hold even in the case when every operator from the semigroup of positive operators is ideal-triangularizable. Namely, R. Drnovšek and others in 2002 constructed an irreducible semigroup of square zero nonzero positive operators on the space $L^p([0, 1))(1 \leq p < \infty)$ with the property that every finite subset of that semigroup is ideal-triangularizable.

We also cannot omit the assumption that the norm is order continuous. Let $\varphi$ be a Banach limit on the atomic Dedekind complete space $l^\infty$ and let $e = (1, 1, \ldots)$ be the order unit of $l^\infty$. Then the positive rank-one operator $T = e \otimes \varphi$ is an idempotent. For every $n \in \mathbb{N}$ we have

$$\varphi_{e_n}(Te_n) = \varphi_{e_n}((e \otimes \varphi)e_n) = \varphi_{e_n}(e)\varphi(e_n) = 0.$$

It can be proved that $T$ is not ideal-triangularizable.
If the lattice is not atomic (3) does not hold even in the case when every operator from the semigroup of positive operators is ideal-triangularizable. Namely, R. Drnovšek and others in 2002 constructed an irreducible semigroup of square zero nonzero positive operators on the space $L^p([0, 1))(1 \leq p < \infty)$ with the property that every finite subset of that semigroup is ideal-triangularizable.

We also cannot omit the assumption that the norm is order continuous. Let $\varphi$ be a Banach limit on the atomic Dedekind complete space $l^\infty$ and let $e = (1, 1, \ldots)$ be the order unit of $l^\infty$. Then the positive rank-one operator $T = e \otimes \varphi$ is an idempotent. For every $n \in \mathbb{N}$ we have

$$\varphi_{e_n}(Te_n) = \varphi_{e_n}((e \otimes \varphi)e) = \varphi_{e_n}(e)\varphi(e_n) = 0.$$  

It can be proved that $T$ is not ideal-triangularizable.
Corollary

Let $I$ be a semigroup of $n \times n$ positive matrices.

1. If every matrix in $I$ has a zero diagonal, then $I$ is ideal-triangularizable and every matrix from $I$ is also nilpotent.

2. If the diagonal of every matrix in $I$ consists only of ones, then $I$ is ideal-triangularizable.
Corollary

Let $\mathcal{I}$ be a semigroup of $n \times n$ positive matrices.

1. If every matrix in $\mathcal{I}$ has a zero diagonal, then $\mathcal{I}$ is ideal-triangularizable and every matrix from $\mathcal{I}$ is also nilpotent.

2. If the diagonal of every matrix in $\mathcal{I}$ consists only of ones, then $\mathcal{I}$ is ideal-triangularizable.
Corollary

Let $\mathcal{S}$ be a semigroup of $n \times n$ positive matrices.

1. If every matrix in $\mathcal{S}$ has a zero diagonal, then $\mathcal{S}$ is ideal-triangularizable and every matrix from $\mathcal{S}$ is also nilpotent.

2. If the diagonal of every matrix in $\mathcal{S}$ consists only of ones, then $\mathcal{S}$ is ideal-triangularizable.
In the preceding corollary we had the situation where the coordinate functionals associated to atoms were constant on a given semigroup. In general, constancy does not imply ideal-triangularizability.

**Example**

For arbitrary $c \in (0, 1)$, the positive matrix

$$A_c = \begin{bmatrix} \frac{c}{\sqrt{c - c^2}} & \sqrt{c - c^2} \\ \sqrt{c - c^2} & 1 - c \end{bmatrix}$$

is an idempotent, and so the coordinate functionals associated to $e_1$ and $e_2$ are constantly equal $c$ and $1 - c$ on the semigroup $\{A_c\}$, respectively. The operator $A_c$ is ideal-irreducible since $B_{e_1}$ and $B_{e_2}$ are not invariant under $A_c$. 
In the preceding corollary we had the situation where the coordinate functionals associated to atoms were constant on a given semigroup. In general, constancy does not imply ideal-triangularizability.

**Example**

For arbitrary $c \in (0, 1)$, the positive matrix

$$A_c = \begin{bmatrix}
c & \sqrt{c - c^2} \\
\sqrt{c - c^2} & 1 - c
\end{bmatrix}$$

is an idempotent, and so the coordinate functionals associated to $e_1$ and $e_2$ are constantly equal $c$ and $1 - c$ on the semigroup $\{A_c\}$, respectively. The operator $A_c$ is ideal-irreducible since $B_{e_1}$ and $B_{e_2}$ are not invariant under $A_c$. 
Theorem

Let $\mathcal{I}$ be a semigroup of positive ideal-triangularizable operators on a Banach lattice with order continuous norm such that every coordinate functional associated to an atom is multiplicative on $\mathcal{I}$. If every operator in $\mathcal{I}$ is a compact or an abstract integral operator, then $\mathcal{I}$ is ideal-triangularizable.
Let $L$ be a Dedekind complete Banach lattice. Denote by $L_r(L)$ the Dedekind complete Riesz space of all regular operators on $L$. It is well known that $L_r(L)$ becomes a Banach lattice algebra with respect to the regular norm defined by $\| T \|_r := \| T \|$. The center $\mathcal{Z}(L)$ is the ideal in $L_r(L)$ generated by the identity operator $I$, i.e.,

$$
\mathcal{Z}(L) = \{ T \in L_r(L) : |T| \leq \lambda I \text{ for some } \lambda \geq 0 \}.
$$

If $T \in \mathcal{Z}(L)$, then the operator norm and the regular norm of $T$ coincide. Since $\mathcal{Z}(L)$ is also a band in $L_r(L)$, we have a band decomposition $L_r(L) = \mathcal{Z}(L) \oplus \mathcal{Z}(L)^d$. Let $P$ be the band projection onto $\mathcal{Z}(L)$. The operator $P$ is a contraction with respect to the operator norm. Schep proved that the component $P(T)$ of a positive operator $T$ in $\mathcal{Z}(L)$ is

$$
P(T) = \inf \left\{ \sum_{i=1}^n P_i TP_i : 0 \leq P_i \leq I, P_i^2 = P_i, \sum_{i=1}^n P_i = I \right\}.
$$
Let $L$ be a Dedekind complete Banach lattice. Denote by $L_r(L)$ the Dedekind complete Riesz space of all regular operators on $L$. It is well known that $L_r(L)$ becomes a Banach lattice algebra with respect to the regular norm defined by $\|T\|_r := \| |T||$. The center $Z(L)$ is the ideal in $L_r(L)$ generated by the identity operator $I$, i.e.,

$$Z(L) = \{ T \in L_r(L) : \|T\| \leq \lambda I \text{ for some } \lambda \geq 0 \}.$$ 

If $T \in Z(L)$, then the operator norm and the regular norm of $T$ coincide. Since $Z(L)$ is also a band in $L_r(L)$, we have a band decomposition $L_r(L) = Z(L) \oplus Z(L)^d$. Let $P$ be the band projection onto $Z(L)$. The operator $P$ is a contraction with respect to the operator norm. Schep proved that the component $P(T)$ of a positive operator $T$ in $Z(L)$ is

$$P(T) = \inf \left\{ \sum_{i=1}^n P_i TP_i : 0 \leq P_i \leq I, P_i^2 = P_i, \sum_{i=1}^n P_i = I \right\}.$$
Let $A$ be the band generated by all atoms in $L$, and let $\mathcal{A} \subseteq A$ be the maximal set of pairwise disjoint atoms of norm one. Given $a \in A$, we denote by $P_a$ the band projection onto the band $\mathcal{B}_a$. Let $T$ be a positive operator on $L$. It can be proved that the operator

\[
\mathcal{D}(T) = \sup \left\{ \sum_{a \in \mathcal{F}} P_a TP_a : \mathcal{F} \text{ is a finite subset of } \mathcal{A} \right\}
\]

exists, since it is a supremum of an increasing net that is bounded from above. We also have $0 \leq \mathcal{D}(T) \leq T$. If $L$ is atomic (i.e., $A = L$), then $\mathcal{D}(T) = P(T)$. For a general $L$ we have

**Proposition**

Let $T$ be a positive operator on $L$. If $P_A$ denotes the band projection onto the band $A$, then $\mathcal{D}(T) = P_A P(T)$. 
Let $A$ be the band generated by all atoms in $L$, and let $\mathcal{A} \subseteq A$ be the maximal set of pairwise disjoint atoms of norm one. Given $a \in A$, we denote by $P_a$ the band projection onto the band $\mathcal{B}_a$. Let $T$ be a positive operator on $L$. It can be proved that the operator

$$D(T) = \sup \left\{ \sum_{a \in \mathcal{F}} P_a TP_a : \mathcal{F} \text{ is a finite subset of } \mathcal{A} \right\}$$

exists, since it is a supremum of an increasing net that is bounded from above. We also have $0 \leq D(T) \leq T$. If $L$ is atomic (i.e., $A = L$), then $D(T) = P(T)$. For a general $L$ we have

**Proposition**

Let $T$ be a positive operator on $L$. If $P_A$ denotes the band projection onto the band $A$, then $D(T) = P_A P(T)$. 
Let $A$ be the band generated by all atoms in $L$, and let $\mathcal{A} \subseteq A$ be the maximal set of pairwise disjoint atoms of norm one. Given $a \in A$, we denote by $P_a$ the band projection onto the band $\mathcal{B}_a$. Let $T$ be a positive operator on $L$. It can be proved that the operator

$$
\mathcal{D}(T) = \sup \left\{ \sum_{a \in \mathcal{F}} P_a T P_a : \mathcal{F} \text{ is a finite subset of } \mathcal{A} \right\}
$$

exists, since it is a supremum of an increasing net that is bounded from above. We also have $0 \leq \mathcal{D}(T) \leq T$. If $L$ is atomic (i.e., $A = L$), then $\mathcal{D}(T) = P(T)$. For a general $L$ we have

**Proposition**

Let $T$ be a positive operator on $L$. If $P_A$ denotes the band projection onto the band $A$, then $\mathcal{D}(T) = P_A P(T)$. 
We extend the operator $\mathcal{D}$ to the operator on the whole space $\mathcal{L}_r(L)$ by defining $\mathcal{D}(T) := P_A \mathcal{P}(T)$ for $T \in \mathcal{L}_r(L)$. This extension is called the *atomic diagonal operator* and the operator $\mathcal{D}(T)$ on $L$ is said to be the *atomic diagonal* of an operator $T \in \mathcal{L}_r(L)$.

**Proposition**

The following assertions hold for the atomic diagonal operator $\mathcal{D}$:

(a) $\mathcal{D}$ is a band projection onto the band

$$\{ T \in \mathcal{L}_r(L) : \|T\| \leq \lambda P_A \text{ for some } \lambda \geq 0 \}$$

in $\mathcal{L}_r(L)$ that can be identified by the center $\mathcal{Z}(A)$;

(b) $\|\mathcal{D}(T)\| \leq \|\mathcal{P}(T)\| \leq \|T\|$ for all $T \in \mathcal{L}_r(L)$;

(c) $\mathcal{P}(T) = \mathcal{D}(T)$ for every positive compact operator $T$ on $L$. 

We extend the operator $D$ to the operator on the whole space $L_r(L)$ by defining $D(T) := PA P(T)$ for $T \in L_r(L)$. This extension is called the atomic diagonal operator and the operator $D(T)$ on $L$ is said to be the atomic diagonal of an operator $T \in L_r(L)$.

**Proposition**

The following assertions hold for the atomic diagonal operator $D$:

(a) $D$ is a band projection onto the band

$$\{ T \in L_r(L) : |T| \leq \lambda PA \text{ for some } \lambda \geq 0 \}$$

in $L_r(L)$ that can be identified by the center $H(A)$;

(b) $\|D(T)\| \leq \|P(T)\| \leq \|T\|$ for all $T \in L_r(L)$;

(c) $P(T) = D(T)$ for every positive compact operator $T$ on $L$. 
Theorem

Let $T$ be a positive power-compact operator on a Banach lattice $L$ with order continuous norm. The following conditions are mutually equivalent:

1. $T$ is ideal-triangularizable;
2. $T - D(T)$ is quasinilpotent;
3. The diagonal entries of $T$ consists precisely of eigenvalues (except maybe zero) of the operator $T$ repeated according to their algebraic multiplicities.
Theorem

Let $T$ be a positive power-compact operator on a Banach lattice $L$ with order continuous norm. The following conditions are mutually equivalent:

1. $T$ is ideal-triangularizable;
2. $T - \mathcal{D}(T)$ is quasinilpotent;
3. The diagonal entries of $T$ consists precisely of eigenvalues (except maybe zero) of the operator $T$ repeated according to their algebraic multiplicities.
Let $T$ be a positive power-compact operator on a Banach lattice $L$ with order continuous norm. The following conditions are mutually equivalent:

1. $T$ is ideal-triangularizable;
2. $T - D(T)$ is quasinilpotent;
3. The diagonal entries of $T$ consists precisely of eigenvalues (except maybe zero) of the operator $T$ repeated according to their algebraic multiplicities.
Theorem

Let $T$ be a positive power-compact operator on a Banach lattice $L$ with order continuous norm. The following conditions are mutually equivalent:

1. $T$ is ideal-triangularizable;
2. $T - D(T)$ is quasinilpotent;
3. The diagonal entries of $T$ consists precisely of eigenvalues (except maybe zero) of the operator $T$ repeated according to their algebraic multiplicities.
Theorem (Drnovšek, Kandić 2009)

Let $E$ be an atomless Banach lattice with order continuous norm and let $\mathcal{S}$ be a semigroup of positive ideal-triangularizable compact operators on $E$. Then $\mathcal{S}$ is ideal-triangularizable.

- So far we have seen that the presence of atoms of a Banach lattice with order continuous norm plays an important role in spectra of ideal-triangularizable positive compact operators.
- Suppose that $S$ and $T$ are positive operators on an atomless Banach lattice with order continuous norm. If one of $S$ and $T$ is compact, then $ST$ and $TS$ are both compact. This implies $\mathcal{D}(ST) = \mathcal{D}(TS) = 0$. 
Theorem (Drnovšek, Kandić 2009)

Let $E$ be an atomless Banach lattice with order continuous norm and let $\mathcal{S}$ be a semigroup of positive ideal-triangularizable compact operators on $E$. Then $\mathcal{S}$ is ideal-triangularizable.

So far we have seen that the presence of atoms of a Banach lattice with order continuous norm plays an important role in spectra of ideal-triangularizable positive compact operators.

Suppose that $S$ and $T$ are positive operators on an atomless Banach lattice with order continuous norm. If one of $S$ and $T$ is compact, then $ST$ and $TS$ are both compact. This implies $\mathcal{D}(ST) = \mathcal{D}(TS) = 0.$
Theorem (Drnovšek, Kandić 2009)

Let $E$ be an atomless Banach lattice with order continuous norm and let $\mathcal{I}$ be a semigroup of positive ideal-triangularizable compact operators on $E$. Then $\mathcal{I}$ is ideal-triangularizable.

- So far we have seen that the presence of atoms of a Banach lattice with order continuous norm plays an important role in spectra of ideal-triangularizable positive compact operators.

- Suppose that $S$ and $T$ are positive operators on an atomless Banach lattice with order continuous norm. If one of $S$ and $T$ is compact, then $ST$ and $TS$ are both compact. This implies $\mathcal{D}(ST) = \mathcal{D}(TS) = 0$. 
**Proposition**

Let $L$ be an atomic Banach lattice $L$ with order continuous norm and $\mathcal{I}$ a semigroup of positive operators on $L$. Then $\mathcal{I}$ is ideal-triangularizable if and only if $\mathcal{D}(ST) = \mathcal{D}(S)\mathcal{D}(T)$ for all $S, T \in \mathcal{I}$.

**Theorem**

Let $\mathcal{I}$ be a semigroup of ideal-triangularizable positive compact operators on a Banach lattice $L$ with order continuous norm such that $\mathcal{D}(ST) = \mathcal{D}(TS)$ for every pair $\{S, T\} \subset \mathcal{I}$. Then the semigroup $\mathcal{I}$ is ideal-triangularizable.
Proposition

Let $L$ be an atomic Banach lattice $L$ with order continuous norm and $\mathcal{S}$ a semigroup of positive operators on $L$. Then $\mathcal{S}$ is ideal-triangularizable if and only if $\mathcal{D}(ST) = \mathcal{D}(S)\mathcal{D}(T)$ for all $S, T \in \mathcal{S}$.

Theorem

Let $\mathcal{S}$ be a semigroup of ideal-triangularizable positive compact operators on a Banach lattice $L$ with order continuous norm such that $\mathcal{D}(ST) = \mathcal{D}(TS)$ for every pair $\{S, T\} \subseteq \mathcal{S}$. Then the semigroup $\mathcal{S}$ is ideal-triangularizable.
Corollary

Let $\mathcal{S}$ be a semigroup of positive $n \times n$ matrices. Suppose that for every matrix $S \in \mathcal{S}$ there exists a permutation matrix $P_S$ (depending on $S$) such that the matrix $P^T_S S P_S$ is upper triangular. If $\mathcal{D}(ST) = \mathcal{D}(TS)$ for every pair $S, T \in \mathcal{S}$, then there exists a permutation matrix $P$ such that every matrix in the semigroup $P^T \mathcal{S} P$ is upper triangular.
**Example**

Let

\[
A = \begin{bmatrix}
0 & 0 & 1 & -1 \\
0 & 0 & -1 & 1 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}
\quad \text{and} \quad
B = \begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 \\
1 & 1 & 0 & 0
\end{bmatrix}.
\]

Then \(A^2 = B^2 = AB = BA = 0\), so that \(\mathcal{S} = \{0, A, B\}\) is a semigroup of ideal-triangularizable matrices such that \(\mathcal{D}(\mathcal{S}) = 0\) for all \(S \in \mathcal{S}\). However, \(\mathcal{S}\) is not ideal-triangularizable, as the diagonal of the matrix

\[
|A| + B = \begin{bmatrix}
0 & 0 & 1 & 1 \\
0 & 0 & 1 & 1 \\
1 & 1 & 0 & 0 \\
1 & 1 & 0 & 0
\end{bmatrix}
\]

is zero, but the matrix is not nilpotent.
Example

Let $e_1, e_2, \ldots, e_n$ be the standard basis vectors of $\mathbb{R}^n$, where $n \geq 3$. Define ideal-triangularizable nilpotent matrices by $A_i = e_i e_{i+1}^T$ for $i = 1, 2, \ldots, n-1$, and $A_n = e_n e_1^T$. Then the collection

$\{ A_1, A_2, \ldots, A_n \}$ has the property that $\mathcal{D}(A_i A_j) = 0$ for all $1 \leq i, j \leq n$. We claim that the collection is not ideal-triangularizable. Assume the contrary. Then the sum $S = A_1 + A_2 + \ldots + A_n$ is ideal-triangularizable. Since all the diagonal entries of $S$ are zero, $S$ must be nilpotent which contradicts the fact that $S^n = I$. 

Main ideas of the proof of the main result

- We prove that $S$ is ideal-reducible since we apply the ideal-triangularization lemma.
- We may assume that $S$ is a closed semigroup which contains every nonnegative multiple of its members.
- If every member of $S$ is quasinilpotent, then $S$ is ideal-reducible.
- Otherwise there exists a nonzero positive operator $S$ in $S$ that is not the identity operator with spectral radius equal to one.
- If the geometric multiplicity of the eigenvalue 1 of $S$ is equal to the algebraic multiplicity, then there is a positive idempotent (not the identity) of finite rank in $S$. This idempotent gives us ideal-reducibility of $S$. 
We prove that $\mathcal{I}$ is ideal-reducible since we apply the ideal-triangularization lemma.

We may assume that $\mathcal{I}$ is a closed semigroup which contains every nonnegative multiple of its members.

If every member of $\mathcal{I}$ is quasinilpotent, then $\mathcal{I}$ is ideal-reducible.

Otherwise there exists a nonzero positive operator $S$ in $\mathcal{I}$ that is not the identity operator with spectral radius equal to one.

If the geometric multiplicity of the eigenvalue 1 of $S$ is equal to the algebraic multiplicity, then there is a positive idempotent (not the identity) of finite rank in $\mathcal{I}$. This idempotent gives us ideal-reducibility of $\mathcal{I}$. 

Main ideas of the proof of the main result
Main ideas of the proof of the main result

- We prove that $\mathcal{S}$ is ideal-reducible since we apply the ideal-triangularization lemma.
- We may assume that $\mathcal{S}$ is a closed semigroup which contains every nonnegative multiple of its members.
- If every member of $\mathcal{S}$ is quasinilpotent, then $\mathcal{S}$ is ideal-reducible.
- Otherwise there exists a nonzero positive operator $S$ in $\mathcal{S}$ that is not the identity operator with spectral radius equal to one.
- If the geometric multiplicity of the eigenvalue 1 of $S$ is equal to the algebraic multiplicity, then there is a positive idempotent (not the identity) of finite rank in $\mathcal{S}$. This idempotent gives us ideal-reducibility of $\mathcal{S}$. 
Main ideas of the proof of the main result

- We prove that $\mathcal{I}$ is ideal-reducible since we apply the ideal-triangularization lemma.
- We may assume that $\mathcal{I}$ is a closed semigroup which contains every nonnegative multiple of its members.
- If every member of $\mathcal{I}$ is quasinilpotent, then $\mathcal{I}$ is ideal-reducible.
- Otherwise there exists a nonzero positive operator $S$ in $\mathcal{I}$ that is not the identity operator with spectral radius equal to one.
- If the geometric multiplicity of the eigenvalue 1 of $S$ is equal to the algebraic multiplicity, then there is a positive idempotent (not the identity) of finite rank in $\mathcal{I}$. This idempotent gives us ideal-reducibility of $\mathcal{I}$. 
Main ideas of the proof of the main result

- We prove that $\mathcal{I}$ is ideal-reducible since we apply the ideal-triangularization lemma.
- We may assume that $\mathcal{I}$ is a closed semigroup which contains every nonnegative multiple of its members.
- If every member of $\mathcal{I}$ is quasinilpotent, then $\mathcal{I}$ is ideal-reducible.
- Otherwise there exists a nonzero positive operator $S$ in $\mathcal{I}$ that is not the identity operator with spectral radius equal to one.
- If the geometric multiplicity of the eigenvalue 1 of $S$ is equal to the algebraic multiplicity, then there is a positive idempotent (not the identity) of finite rank in $\mathcal{I}$. This idempotent gives us ideal-reducibility of $\mathcal{I}$. 
Otherwise there exists a square zero nonzero positive operator $M$ in $\mathcal{S}$. It can be proved that the semigroup ideal in $\mathcal{S}$ generated by $M$ consists of quasinilpotent operator so that $\mathcal{S}$ is again ideal-reducible.
Thank you for your attention.