Mean Ergodic Theorems on Norming Dual Pairs

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based on joint work with Moritz Gerlach

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X Banach Space, $T \in \mathscr{L}(X)$ *power bounded,* $A_n := \frac{1}{n}\sum_{k=0}^{n-1} T^k$.

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1. $\|\cdot\|$-$\lim_{n\to\infty} A_n x$ *exists for all $x \in X$, i.e. T is mean ergodic.*
2. $A_n x$ *has a $\sigma(X, X^*)$-cluster point for all $x \in X$.*
3. $\mathrm{fix}(T)$ *separates* $\mathrm{fix}(T^*)$*.*
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- Typical space not to work on: $C_b(\Omega)$, $\mathscr{M}(\Omega)$ Ω Polish.

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- Mean ergodic theorems on locally convex spaces: Require *equicontinuity*!

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- We consider $X = C_b(\Omega)$ and $Y = \mathscr{M}(\Omega)$ in duality via

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- $\rightsquigarrow$ Markov chains/processes on Polish spaces.

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- Extension to more general semigroups and average schemes.

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We give an example for the latter. Put
$K_n := \{0, 1, \ldots, n\} \times \{n^{-1}\}$ $\quad K_0 := \mathbb{N} \times \{0\}$ $\quad \Omega := \bigcup_{n=0}^{\infty} K_n$.

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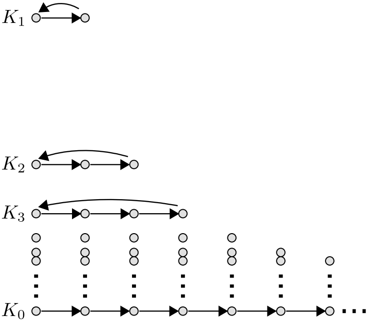
$K_n := \{0, 1, \ldots, n\} \times \{n^{-1}\}$ $\quad K_0 := \mathbb{N} \times \{0\}$ $\quad \Omega := \bigcup_{n=0}^{\infty} K_n$.
We put $X = C_b(\Omega)$ $\quad Y = \mathscr{M}(\Omega)$. $\quad T \in \mathscr{L}(X, \sigma)$ $\quad$ is given by
$Tf = f \circ \varphi$, where φ acts as follows.

Counterexamples cont'd



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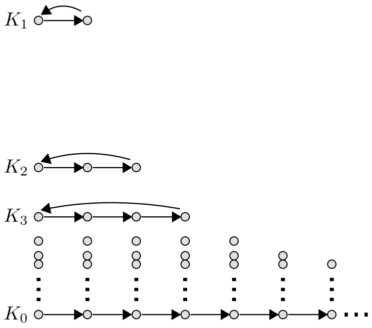

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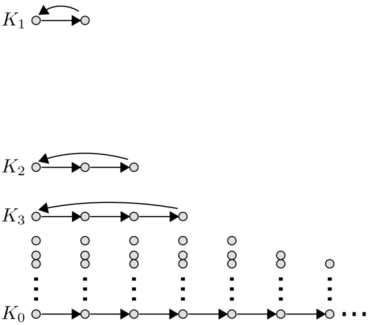

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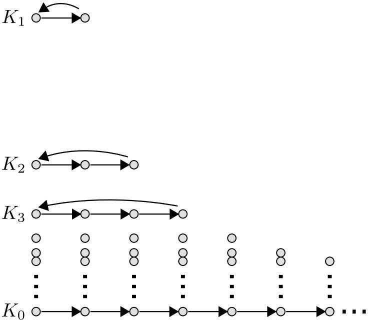

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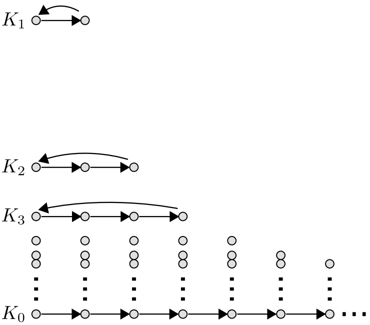

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But $\delta_{(0,0)} \notin \operatorname{fix}(T') \oplus \overline{\operatorname{rg}(I - T')}^{\sigma'}$.

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Lemma

(RC) and $\{A_n' \delta_x\}$ tight for all $x \in \Omega$ and T Markovian
$\Rightarrow \{A_n : n\}$ is β_0-equicontinuous.

Main Result II

Theorem

Let T be Markovian and assume (RC). Equivalent:

- β_0-$\lim_{n\to\infty} A_n' f$ *exists for all* $f \in C_b(\Omega)$.
- $A_n'\delta_x$ *has a* σ'*-cluster point for every* $x \in \Omega$.
- $\mathrm{fix}(T')$ *separates* $\mathrm{fix}(T)$.
- $\mathscr{M}(E) = \mathrm{fix}(T) \oplus \overline{\mathrm{rg}(I - T')}^{\sigma'}$.

The End

Thank you for your attention.