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Algebraic Band Preserving Operators

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Let E and F be Archimedean vector lattices.

Definition 1. An linear operator $T : E \to F$ is said to be band preserving if $|Tu| \wedge |v| = 0$ for all u and v in E with $|u| \wedge |v| = 0$.

Definition 2. A linear operator $T : E \to F$ is called order bounded if for every $v \in E_+$ there exists $u \in F_+$ such that $|T(w)| \leq u$ in F whenever $|w| \leq v$ in E.

An order bounded band preserving operator on E is an orthomorphism on E.

The Wickstead Problem

- Is any band preserving linear operator in a vector lattice automatically order bounded?

The question was raised in
Wickstead A. Representation and duality of multiplication operators on Archimedean Riesz spaces// Composition Math. **35**(3), 1977, p. 225—238.

Recall that a universally complete vector lattice is a vector lattice, which is Dedekind complete and laterally complete (any set of pairwise disjoint elements is bounded).

The Wickstead Problem

The first example of a band preserving unbounded operator:

Abramovich Yu. A., Veksler A. I., Koldunov A. V. Operators preserving disjointness // Dokl. Academ. Nauk USSR **248** (1979), 1033–1036.

A characterisation of universally complete vector lattices in which all band preserving operators are order bounded was given in:

Abramovich Yu. A., Veksler A. I., Koldunov A. V. Disjointness preserving operators, their continuity and multiplicative representation // Linear Operators and Their Applications, Leningrad Ped. Inst., Leningrad, (1981). [in Russian]

McPolin P. and Wickstead A. The order boundedness of band preserving operators on uniformly complete vector lattices // Math. Proc. Cambridge Philos. Soc., **97**(3) (1985), 481–487

d-Basis

Let E be a vector lattice with a cofinal family of projection bands.

Definition 3. A collection of elements $\{e_\gamma : \gamma \in \Gamma\} \subset E$ is said to be *d-independent*, if for each projection band B in E the set $\{\rho_B e_\gamma : \rho_B e_\gamma \neq 0, \gamma \in \Gamma\}$ is linearly independent.

Any maximal (by inclusion) set of d-independent vectors is called *d-basis*.

Consider a vector lattice E and let $\mathcal{E} := \{e_\gamma, \gamma \in \Gamma\}$ be a fixed d-basis. Then for each element $x \in E$ there exists a full collection $\rho_\xi \quad (\xi \in \Xi)$ of pairwise disjoint band projections (depending on x) such that for each index ξ the set $\Gamma_\xi = \{\gamma \in \Gamma : \rho_\xi e_\gamma \neq 0\}$ is finite and the elements $\rho_\xi x$ are the linear combination of these linearly independent projections $\rho_\xi e_\gamma$ with $\gamma \in \Gamma_\xi$. So for each $x \in E$ the following representation holds:

$$x = \sum_{\xi} \sum_{\gamma \in \Gamma} \alpha_\gamma^\xi \rho_\xi e_\gamma,$$

where α_γ^ξ are some scalars (depending on x), such that for each ξ only a finite number of coefficients α_γ^ξ may be nonzero. This expression is called a d-expansion of x with respect to d-basis $\{e_\gamma\}$.

Locally One-dimensional Vector Lattices

Definition 4. A universally complete vector lattice E is called locally one-dimensional if $\{\mathbf{1}\}$ is a d-basis of E.

Theorem 1. Assume, that E is a universally complete vector lattice. Then the following assertions are equivalent:

(1) E is locally one-dimensional.

(2) every band preserving linear operator in E is order bounded.

So the Wickstead problem in a class of universally complete vector lattices was reduced to the description of locally one-dimensional vector lattices. And this led to another problem:

- Is the class of locally one-dimensional vector lattices coincident with the class of atomic vector lattices?

Abramovich Yu., Wickstead A. The regularity of order bounded operators into $C(K)$ // Quart. J. Math. Oxford, Ser. 2, **44** (1993), 257–270.

Definition 5. An element $x \in E_+$ is called an atom or discrete if $[0, x] = [0, 1]x$ in other words: if from $0 \leq y \leq x$ follows that $y = \lambda x$ for some $0 \leq \lambda \leq 1$.

A vector lattice is called atomic or discrete if for every element $0 \neq y \in E_+$ there exists a discreet element $x \in E$ such that $0 < x \leq y$.

Gutman A. Locally one-dimensional K-spaces and σ-distributive Boolean algebras // Siberian Adv. Math, **5**(2), 1995, p. 99–121.

Theorem 2. There is a purely nonatomic locally one-dimensional universally complete vector lattice.

Theorem 3. A universally complete vector lattice is locally one-dimensional if and only if it's base is σ-distributive.

σ-Distributive Boolean Algebras

Recall that the base $\mathfrak{B}(E)$ of a vector lattice E is the complete Boolean algebra of all bands in E.

Definition 6. A Boolean σ-algebra B is called σ-distributive if for every double sequence $(b_{n,m})_{n,m\in\mathbf{N}}$ in B the following holds:

$$\bigvee_{n\in\mathbf{N}} \bigwedge_{m\in\mathbf{N}} b_{n,m} = \bigwedge_{\varphi\in\mathbf{N}^{\mathbf{N}}} \bigvee_{n\in\mathbf{N}} b_{n,\varphi(n)}$$

Definition 7. A linear operator $T : E \to E$ is said to be algebraic if there exists a non-zero annihilator polynomial φ, that is $\varphi(T) = 0$.

An annihilator polynomial of minimal degree is called minimal polynomial of operator T and is denoted by φ_T.

Simple examples of algebraic operators:

- A projection P (an idempotent operator: $P^2 = P$) on E. If $P \neq 0, 1$, then it's minimal polynomial is $\varphi_P(\lambda) = \lambda^2 - \lambda$;
- A nilpotent operator ($S^m = 0$ for some $m \in \mathbf{N}$) on E. And it's minimal polynomial is $\varphi_S(\lambda) = \lambda^k$, where $k \leq m$.

The Purpose of This Work

- When are all algebraic band preserving operators order bounded in a vector lattice?

In other words, we want to examine the Wickstead problem for the class of algebraic operators.

Strongly Diagonal Operators

Definition 8. We call T a strongly diagonal operator on E if there exist pairwise disjoint band projections $\rho_1, \dots, \rho_m$ and real numbers $\alpha_1, \dots, \alpha_m$ such that $T = \alpha_1\rho_1 + \dots + \alpha_m\rho_m$.

In the equality above, we may assume that $\rho_1 + \dots + \rho_m = I_E$ and that $\alpha_1, \dots, \alpha_m$ are pairwise different.

In particular, every strongly diagonal operator on E is an orthomorphism.

Boulabiar K., Buskes G., Sirotkin G. Algebraic order bounded disjointness preserving operators and strongly diagonal operators // Integral Equations and Operator Theory, 54, p. 9–31, 2006.

Theorem 4. Let E be an Archimedean vector lattice. An operator T on E is strongly diagonal if and only if the following conditions hold:

(1) T is order bounded.
(2) T is band preserving.
(3) T is algebraic.

The following result is proved in
Abramovich Yu. A., Kitover A. K. Inverses of disjointness preserving operators // Memoirs of the American Mathematical Society,vol. 143, №679, 2000.

Theorem 5. If E is a universally complete vector lattice, then for each non-zero band B there is a non-zero band $B_0 \subseteq B$ such that B_0 has a d-basis consisting of weak order units in B_0.

Theorem 6. Let E be a universally complete vector lattice. Then the following assertions are equivalent:

(1) The Boolean algebra $\mathfrak{B}(E)$ is σ-distributive.

(2) Every algebraic band preserving operator in E is order bounded.

(3) Every algebraic band preserving operator in E is strongly diagonal.

(4) Every band preserving projection in E is a band projection.

Proof: (1) ⇒ (2) ⇒ (3) ⇒ (4)

(1) ⇒ (2) Gutman (Theorem 3).

(2) ⇒ (3) Boulabiar–Buskes–Sirotkin (Theorem 4).

(3) ⇒ (4) Let P be a band preserving projection in E. Then P is algebraic, and thus strongly diagonal by (3):

$$P = \sum_{i=1}^{m} \alpha_i \rho_i \quad (\rho_i \perp \rho_j,\ i \neq j).$$

Multiplying both parts of this equality by ρ_i, we get:

$$P\rho_i = \alpha_i \rho_i \quad (i = 1, \dots, m).$$

Since $P\rho_i$ is a projection, we have: $\alpha_i^2 \rho_i = \alpha_i \rho_i \Rightarrow \rho_i(\alpha_i^2 - \alpha_i) = 0$. Thus, if $\rho_i \neq 0$ then either $\alpha_i = 1$ or $\alpha_i = 0$ and P is a band projection as the sum of pair-wise disjoint band projections.

Proof: (4) $\Rightarrow$ (1)

(4) $\Rightarrow$ (1) By contradiction: We assume that $\mathfrak{B}(E)$ is not σ-distributive and construct a band preserving projection which is not a band projection.

Let $\mathcal{E} = \{e_\gamma\}_{\gamma\in\Gamma}$ be an arbitrary d-basis in E. For an element $x \in E$ there exists a full collection of pairwise disjoint band projections ρ_ξ, such that the representation holds:

$$x = \sum_{\xi} \sum_{\gamma\in\Gamma} \alpha_\gamma^\xi \rho_\xi e_\gamma,$$

where the set $\{\gamma \in \Gamma : \alpha_\gamma^\xi \neq 0\}$ is finite for all ξ. By Theorem 5 we may assume that $\mathbf{1} \in \mathcal{E}$. Fix $\mathbf{1} \neq \gamma_0 \in \Gamma$ and define a band preserving projection $P : E \to E$ by

$$Px := \sum_{\xi} \alpha_{\gamma_0}^\xi \rho_\xi e_{\gamma_0} \quad (x \in E). \square$$

Band Preserving Nilpotent Operators

Theorem 7. Let E be a universally complete vector lattice. Then the following assertions are equivalent:

(1) The Boolean algebra $\mathfrak{B}(E)$ is σ-distributive.

(2) Every band preserving nilpotent operator in E is order bounded.

(3) Every band preserving nilpotent operator in E is strongly diagonal.

(4) Every band preserving nilpotent operator in E is trivial.

Proof: (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4)

(1) $\Rightarrow$ (2) Gutman (Theorem 3).
(2) $\Rightarrow$ (3) Boulabiar–Buskes–Sirotkin (Theorem 4).
(3) $\Rightarrow$ (4) Consider a band preserving nilpotent operator T. by (3) T is strongly diagonal: $T = \sum_{i=1}^{s} \alpha_i P_i$, where $P_i \perp P_j$ ($i \neq j$; $i, j = 1, \dots, s$. Obviously,

$$T^N = \Big(\sum_{i=1}^{s} \alpha_i P_i \Big)^N = \sum_{i=1}^{s} \alpha_i^N P_i = 0.$$

As P_i are pair-wise disjoint, we have from the last equality that $\alpha_i = 0$ for all $i = 1, \dots s$ and thus $T = 0$.

Proof: (4) $\Rightarrow$ (1)

(4) $\Rightarrow$ (1) By contradiction: We assume that $\mathfrak{B}(E)$ is not σ-distributive and construct a non-trivial band preserving operator T with $T^m = 0$, $m \in \mathbf{N}$. By Theorem 5 we can also assume that there exists a d-basis in E consisting of weak order units.

Again we have a representation of $x \in E$:

$$x = \sum_{\xi} \sum_{\gamma \in \Gamma} \lambda_{\gamma}^{\xi} \rho_{\xi} e_{\gamma},$$

where the set $\{\gamma \in \Gamma : \alpha_{\gamma}^{\xi} \neq 0\}$ is finite for all ξ. Fix $e_1, \ldots, e_{m+1} \in \mathcal{E}$ and put

$$T(e_i) = e_{i+1} \ (i = 1, \ldots, m); \quad T(e) = 0, \text{ if } e \neq e_i \ (i = 1, \ldots, m).$$

Proof: (4) $\Rightarrow$ (1)

This means that the operator $T : E \to E$ is defined as follows:

$$T(x) = \sum_{\xi \in \Xi} \sum_{i=1}^{m+1} \alpha_\xi^i \rho_\xi(e_{i+1}).$$

T is a linear band preserving (by construction) operator associated to a square $(m+1) \times (m+1)$-matrix A:

$$A = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ 0 & 0 & 0 & \dots & 0 \end{pmatrix}$$

It follows that $T^m = 0$. $\square$

Theorem 8. Let E be a universally complete vector lattice. The following assertions are equivalent:

(1) The Boolean algebra $\mathfrak{P}(E)$ is σ-distributive.

(2) Every algebraic band preserving operator in E is order bounded.

(3) Every algebraic band preserving operator in E is strongly diagonal.

(4) Every band preserving projection in E is a band projection.

(5) Every band preserving nilpotent operator in E is order bounded.

(6) Every band preserving nilpotent operator in E is strongly diagonal.

(7) Every band preserving nilpotent operator in E is trivial.

Gutman A.E., Kusraev A.G., Kutateladze S.S.
The Wickstead Problem // Siberian Electronic Math. Reports. 2008. V. 5. P. 293-333.

Theorem 9. The vector lattice $L^0(\Omega, \Sigma, \mu)$ is locally one-dimensional if and only if it is atomic ($\equiv$ the associated Boolean algebra $\mathfrak{B}(\Omega, \Sigma, \mu)$ is atomic).

Theorem 10. For $L^0(\Omega, \Sigma, \mu)$ with atomless $\mathfrak{B}(\Omega, \Sigma, \mu)$ the following statements hold:

(1) there exists a nontrivial band preserving projection in $L^0(\Omega, \Sigma, \mu)$, which is not order bounded.

(2) there exists a nontrivial band preserving nilpotent operator in $L^0(\Omega, \Sigma, \mu)$, which is not order bounded.

The end

THANK YOU FOR ATTENTION!