An order-theoretic approach to stochastic processes

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- The theory of Riesz spaces (vector lattices) is a generalisation of measure-and-integration theory.
- Is it possible to develop a theory of stochastic processes in this setting; i.e., without the use of probability?

Wish list for time and space

Which time setting?

- discrete time
- continuous time

Which space setting?

- Riesz space
- Banach lattice
- Bochner space
- space of integrably bounded functions
- ...

Wish list for a theory of stochastic processes

- conditional expectation
- filtration
- martingale / sub / supermartingale
  - convergence
  - decompositions
- stopping times
- independence
- ...
- Markov processes
- Brownian motion
- stochastic integral
  - Girsanov theorem
  - martingale representation theoerem
  - Itô calculus (?)

Outline

- Discrete time Riesz space stochastic processes
- Continuous time stochastic processes
- Discrete time Banach space/lattice stochastic processes

Discrete time
Riesz space stochastic processes

Conditional expectations

Let (Ω, Σ, μ) be a probability space and Σ_0 a sub-σ-algebra of Σ.

The conditional expectation operator $\mathbb{E}[\,\cdot\,|\Sigma_0]$ associated with Σ_0 is a map $\mathbb{E}[\,\cdot\,|\Sigma_0] : L^1(\Sigma, \mu) \to L^1(\Sigma_0, \mu)$ which is

(1) linear;

(2) positive;

(3) $\mathbb{E}[\mathbf{1}|\Sigma_0] = \mathbf{1}$, where $\mathbf{1}$ is the function with value 1 almost everywhere;

(4) $\mathbb{E}[\mathbb{E}[f|\Sigma_0]|\Sigma_0] = \mathbb{E}[f|\Sigma_0]$; i.e., $\mathbb{E}[\,\cdot\,|\Sigma_0]$ is idempotent;

(5) if $f_n \uparrow f$, then $\mathbb{E}[f_n|\Sigma_0] \uparrow \mathbb{E}[f|\Sigma_0]$;
i.e., $\mathbb{E}[\,\cdot\,|\Sigma_0]$ is order continuous.

E is a Riesz space (Dedekind complete with a weak order unit e).

Definition

If $T : E \to E$ is a positive linear projection which is order continuous and the range $\mathcal{R}(T)$ of T is a Dedekind complete vector lattice and a sublattice of E (and $T(e) = e$), then T is called a conditional expectation on E.

Filtration

If (Σ_i) is a filtration; i.e., (Σ_i) an increasing sequence of sub-σ-algebras of Σ, then

- $\mathbb{E}\left[\mathbb{E}[f|\Sigma_i]\,|\Sigma_j\right] = \mathbb{E}[f|\Sigma_i] = \mathbb{E}\left[\mathbb{E}[f|\Sigma_j]\,|\Sigma_i\right]$ for all $i \leq j$.

Thus, $(\mathbb{E}[\,\cdot\,|\Sigma_i])_{i\in\mathbb{N}}$ is a commuting sequence of operators.

Definition

A filtration $(T_i)_{i\in\mathbb{N}}$ on E is a sequence of conditional expectations on E with

$$T_iT_j = T_{i\wedge j}.$$

(and $T_j(e) = e$ for all $j \in \mathbb{N}$).

Martingales/sub- and supermartingales

If $(\Omega, \Sigma, (\Sigma_i)_{i\in\mathbb{N}}, \mu)$ is a filtered probability space, $(f_i, \Sigma_i)_{i\in\mathbb{N}}$ is called a martingale (submartingale) [supermartingale] if $f_i \in L^1(\Sigma_i, \mu)$ for each $i \in \mathbb{N}$ and

$$\mathbb{E}[f_j|\Sigma_i] = \ (\geq)\ [\leq] f_i \text{ for all } j \geq i.$$

Definition

The pair $(f_i, T_i)_{i\in\mathbb{N}}$ is a martingale (submartingale) [supermartingale] in the Riesz space E, if (T_i) is a filtration, $f_i \in \mathcal{R}(T_i)$ (the range of T_i) for each $i \in \mathbb{N}$ and

$$T_i f_j = \ (\geq)\ [\leq] f_i \text{ for all } i \leq j.$$

Doob's decomposition

Theorem

Let (f_i, T_i) *be a submartingale (supermartingale) and*

$$\begin{aligned} A_j &= \sum_{i=1}^{j-1} T_i(f_{i+1} - f_i) \\ M_j &= f_j - A_j \end{aligned}$$

for all $j \in \mathbb{N}$. *The decomposition*

$$f_i = M_i + A_i \text{ for all } i \in \mathbb{N},$$

is the unique decomposition of (f_i, T_i) *with* (M_j, T_j) *a martingale,* (A_j) *positive and increasing (negative and decreasing),* $A_1 = 0$ *and* $A_{j+1} \in \mathcal{R}(T_j)$ *for all* $j \in \mathbb{N}$.

Martingale convergence

The main tool required to prove the next result is the Upcrossing theorem (continuous time, Koos Grobler – Positivity 7).

Theorem

Let (f_i, T_i) be a sub (or super) martingale in a Dedekind complete Riesz space E with weak order unit and each T_i is strictly positive (i.e. T_i is positive and $\{f \in E : T_i(|f|) = 0\} = \{0\}$). If there exists $g \in E^+$ such that $|f_i| \leq g$ for all $i \in \mathbb{N}$, then (f_i) is order convergent and the order limit $f_\infty \in E$ is given by

$$\limsup f_i = f_\infty = \liminf f_i.$$

Definition

Let E be a Dedekind complete Riesz space and T a strictly positive conditional expectation on E. The space E is universally complete with respect to T if for each increasing net (f_α) in E^+ with (Tf_α) order bounded, we have that (f_α) is order convergent.

Theorem

Let (f_i, T_i) be a sub (or super) martingale in a Dedekind complete, T_1-universally complete Riesz space E with weak order unit and in which the operators T_1 is strictly positive. If there exists $g \in E^+$ such that $T_1|f_i| \leq g$ for all $i \in \mathbb{N}$, then (f_i) is order convergent and the order limit $f_\infty \in E$ is given by

$$\limsup f_i = f_\infty = \liminf f_i.$$

Stopping times

Let (Σ_i) be a filtration on the probability space (Ω, Σ, μ).

- A stopping time adapted to (Σ_i) is a map $\tau : \Omega \to \mathbb{N} \cup \{\infty\}$ such that

$$\tau^{-1}(\{1, ..., n\}) \in \Sigma_n \text{ for each } n \in \mathbb{N}.$$

- The stopping time τ is said to be bounded if there exists $n \in \mathbb{N}$ such that $\tau(x) \leq n$ almost everywhere on Ω; i.e., up to measure zero, $\tau^{-1}(\{1, ..., n\}) = \Omega$.

Stopping times

Definition

Let (T_i) be a filtration on E.

- A stopping time $\mathcal{P} = (P_i)$ adapted to the filtration (T_i) is an increasing sequence of positive order continuous linear projections (P_i) with $\mathcal{R}(P_i)$ a Dedekind complete Riesz subspace of E, and
  - $P_0 = 0$,
  - $P_i \leq id_E$ for all i,
  - $T_j P_i = P_i T_j$ for all $i \leq j$.
- The stopping time (P_i, T_i) is bounded if there exists N so that $P_n = id_E$ for all $n \geq \mathbb{N}$.

Stopped processes

Definition

Let $\mathcal{P} = (P_i)$ be a bounded stopping time adapted to the filtration (T_i). For each $(f_i) \subseteq E$ with $f_i \in \mathcal{R}(T_i)$ for all $i \in \mathbb{N}$, we define the stopped process $(f_P, T_\mathcal{P})$ by

$$
\begin{aligned}
f_\mathcal{P} &= \sum_{i=1}^{\infty} (P_i - P_{i-1}) f_i, \\
T_\mathcal{P} f &= \sum_{i=1}^{\infty} (P_i - P_{i-1}) T_i f, \quad f \in E.
\end{aligned}
$$

Order

Definition

Let $\mathcal{P}$ and $\mathcal{S}$ be stopping times adapted to the filtration (T_i). Define

$$\mathcal{S} \leq \mathcal{P} \iff P_i \leq S_i \text{ for all } i \in \mathbb{N}.$$

Hunt's Theorem / Doob's optional sampling theorem

Theorem

Let $\mathcal{S} \leq \mathcal{P}$ be bounded stopping times adapted to the filtration (T_i). Then $(f_i, T_i)_{i\in\mathbb{N}}$ a martingale (submartingale) [supermartingale] in E if and only if

$$T_{\mathcal{S}} f_{\mathcal{P}} = (\geq)\,[\leq]\, = f_{\mathcal{S}};$$

i.e., $(f_{\mathcal{P}}, T_{\mathcal{P}})$ is a (sub, super) martingale over the sequence of all stopping times adapted to the filtration $(T_{\mathcal{P}})$ and indexed by the partially ordered set of all bounded stopping times adapted to (T_i).

Remarks on pre-existing references

- DeMarr, R., A martingale convergence theorem in vector lattices. Can. J. Math. 18, 424 – 432 (1966)
- Stoica, G., Vector valued quasi-martingales. Stud. Cerc. Mat. 42, 73 – 79 (1990)
- Stoica, G., The structure of stochastic processes in normed vector lattices. Stud. Cerc. Mat. 46, 477 – 486 (1994)

Continuous time
Riesz space stochastic processes

Markov processes

Jessica Vardy – Positivity 6

- VARDY, JESSICA J.; WATSON, BRUCE A., Markov processes on Riesz spaces. Positivity 16 (2012), no. 2, 373 – 391.
- VARDY, JESSICA J.; WATSON, BRUCE A. Erratum to: Markov processes on Riesz spaces. Positivity 16 (2012), no. 2, 393.

Doob-Meyer decomposition

Doob-Meyer (Koos Grobler – Positivity 6).

- Grobler J.J., Continuous stochastic processes in Riesz spaces: the Doob-Meyer decomposition, Positivity 14 (2010), 731 – 751

Theorem

Let $(\mathbb{F}_t, \mathcal{F}_t)$ be a filtration on a perfect Riesz space E and assume E is $\mathbb{F}_0$-universally complete. Any tractable class DL submartingale $X = (X_t)$ has a decomposition

$$X_t = M_t + A_t, \ \ t \in [0, \infty),$$

where $M = \{M_t, \mathcal{F}_t, t \geq 0\}$ is a right continuous martingale and $A = \{A_t, \mathcal{F}_t, t \geq 0\}$ and increasing predictable process. The decomposition is unique.

Doob's optional sampling theorem for continuous time:

- GROBLER, J. J., Doob's optional sampling theorem in Riesz spaces. Positivity 15 (2011), 617 – 637.

Grobler – Positivity 7.

- GROBLER, J.J., Jensen's and martingale inequalities in Riesz spaces. Indagationes Mathematica, in press (2013).

Stochastic integral

The discrete time stochastic integral exists in the Riesz space setting:

- C.C.A. Labuschagne, B.A. Watson, Discrete time stochastic integrals in Riesz spaces, Positivity 14 (2010), 859 – 875.

The continuous time case is open – work in progress.

Discrete time
Banach lattice stochastic processes

- Troitsky, V., Martingales in Banach lattices. Positivity 9, 437 – 456 (2005)

Banach spaces/lattices

E and F will denote Banach lattices and X and Y will denote Banach spaces.

Definition

- Let (T_i) be a sequence of contractive linear projections on Y and $T_{i \wedge j} = T_i T_j$ for each $i, j \in \mathbb{N}$. Then (T_i) is called a BS-filtration on Y.
- If (T_i) is a BS-filtration on E with the property that each $T_i \geq 0$ and $\mathcal{R}(T_i)$ is a (closed) Riesz subspace of E, then (T_i) will be called a BL-filtration on E.

Definition

Let (T_i) a BS-filtration on Y. Let

- $\mathcal{M}(Y, T_i) = \{(f_i, T_i) \text{ a martingale in } Y : \sup_i \|f_i\| < \infty\}$ and

$$\|(f_i, T_i)\| = \sup_i \|f_i\| \text{ for all } (f_i, T_i) \in \mathcal{M}(Y, T_i).$$

- $\mathcal{M}_{\mathrm{nc}}(Y, T_i) = \{(f_i, T_i) \in \mathcal{M}(Y, T_i) : (f_i) \text{ is norm convergent in } Y\}$.

Convergent martingales

The space $\mathcal{M}_{nc}(X, T_i)$ of norm convergent martingales on X has the following description:

Theorem

Let be a BS-filtration on X. *Then* $L : \mathcal{M}_{nc}(X, T_i) \to \overline{\bigcup_{i=1}^{\infty} \mathcal{R}(T_i)}$, *defined by* $L\left((f_i, T_i)\right) = \lim_i f_i$, *is a surjective isometry.*

Definition

If (T_i) is a BL-filtration on a Banach lattice E, $T_i \geq 0$ and (f_i, T_i) is a martingale, define

$$(f_i, T_i) \geq 0 \iff f_i \geq 0 \text{ for all } i \in \mathbb{N}.$$

Theorem

Let (T_i) be a BL-filtration on E. If $L : \mathcal{M}_{nc}(E, T_i) \to \overline{\bigcup_{i=1}^{\infty} \mathcal{R}(T_i)}$ is defined by $L((f_i, T_i)) = \lim_i f_i$, then $\mathcal{M}_{nc}(E, T_i)$ is a Banach lattice and $L : \mathcal{M}_{nc}(E, T_i) \to \overline{\bigcup_{i=1}^{\infty} \mathcal{R}(T_i)}$ is a surjective Riesz isometry.

Stopping times - again

Definition

Let X be a Banach space. A collection $\mathcal{T} \subset \mathcal{L}(X)$ is said to be R-bounded if there exists a constant $M > 0$ such that

$$\left\| \sum_{i=1}^{n} r_i \otimes T_i x_i \right\|_{L^2(\mu, X)} \leq M \left\| \sum_{i=1}^{n} r_i \otimes x_i \right\|_{L^2(\mu, X)}$$

holds for all $(T_i)_{i=1}^n \subset \mathcal{L}(X)$, $(x_i)_{i=1}^n \subset X$ and $n \in \mathbb{N}$. Here, the sequence (r_i) denotes the Rademacher functions.

Theorem

Let E be an order continuous Banach lattice and (T_i) a positive R-bounded BS-filtration on E with R-bound M. If $\mathcal{P} = (P_i)$ is a (not necessarily bounded) stopping time adapted to (T_i), then

$$T_{\mathcal{P}} f := \sum_{i=1}^{\infty} (P_i - P_{i-1}) T_i f$$

for all $f \in E$ defines a bounded linear projection with $\|T_{\mathcal{P}}\| \leq M$. Consequently, $\sup_{\mathcal{P} \in \mathbb{T}^*} \|T_{\mathcal{P}}\| \leq M$.

This yields a Doob's optional sampling theorem for unbounded stopping times.

Discrete time
Bochner space stochastic processes

- So far, we have exploited the order structure of the Riesz space / Banach lattice
- What about $L^p(\mu, X)$, where X is a Banach space?

If Σ_1 is a sub σ-algebra of Σ, the conditional expectation of $f \in L^p(\mu, X)$ relative to Σ_1, denoted by $\mathbb{E}(f \mid \Sigma_1)$, is a Σ_1-measurable element of $L^p(\mu, X)$ which is given by

$$\int_A \mathbb{E}(f \mid \Sigma_1) d\mu = \int_A f d\mu \quad \text{for all } A \in \Sigma_1.$$

$L^p(\mu, Y)$ as a tensor product

For $1 \leq p < \infty$, define a bilinear mapping $\psi : L^p(\mu) \times Y \to L^p(\mu, Y)$ by

$$\psi(f, y)(\omega) = f(\omega)y \quad \text{for all } \omega \in \Omega.$$

Then the induced linear map $\psi^{\otimes} : L^p(\mu) \otimes Y \to L^p(\mu, Y)$ is described by

$$\psi^{\otimes}(f \otimes y)(\omega) = f(\omega)y \quad \text{for all } \omega \in \Omega$$

and is injective.

- Thus, $L^p(\mu, Y)$ contains a copy of $L^p(\mu) \otimes Y$ and we may induce the Bochner norm on $L^p(\mu) \otimes Y$.
- The normed space $(L^p(\mu) \otimes Y, \| \cdot \|_p)$ is denoted $L^p(\mu) \otimes_{\Delta_p} Y$.

Let $S_p(\mu)$ denote the real-valued simple functions in $L^p(\mu)$, we write

$$S_p(\mu) \otimes Y := \left\{ \sum_{k=1}^{n} \chi_{A_k} \otimes y_k : n \in \mathbb{N},\ \chi_{A_k} \text{ integrable},\ y_k \in Y \right\}$$

and let

$$S_p(Y) := \left\{ \sum_{k=1}^{n} y_k \chi_{A_k} : n \in \mathbb{N},\ \chi_{A_k} \text{ integrable},\ y_k \in Y \right\}$$

denote the Y-valued simple functions in $L^p(\mu, Y)$.

Then, the restricted map

$$\psi^{\otimes} : S_p(\mu) \otimes Y \to S_p(Y)$$

defines a one-to-one, norm preserving correspondence between $S_p(\mathbb{R}) \otimes Y$ and $S_p(Y)$. Hence

$$\psi^{\otimes}(S_p(\mu) \otimes Y) = S_p(Y) \subset \psi^{\otimes}(L^p(\mu) \otimes_{\Delta_p} Y) \subset L^p(\mu, Y).$$

Since $S_p(\mu)$ is dense in $L^p(\mu)$ and $S_p(Y)$ is dense in $L^p(\mu, Y)$ it follows, by taking completions, that
$L^p(\mu)\widetilde{\otimes}_{\Delta_p} Y$ is isometrically isomorphic to $L^p(\mu, Y)$.

Conditional expectation – again

We recall the construction of $\mathbb{E}(f \mid \Sigma_1)$. Let Σ_1 be a sub σ-algebra of Σ. Define $\mathbb{E}(\,\cdot \mid \Sigma_1) : S_p(X) \to S_p(X)$ by

$$\mathbb{E}\left(\sum_{i=1}^{n} \chi_{A_i} \otimes x_i \mid \Sigma_1\right) = (\mathbb{E}(\,\cdot \mid \Sigma_1) \otimes id_X)\left(\sum_{i=1}^{n} \chi_{A_i} \otimes x_i\right),$$

where $\mathbb{E}(\chi_{A_i} \mid \Sigma_1)$ denotes the conditional expectation of $\chi_{A_i} \in L^p(\mu)$.
By Jensen's inequality, it can be shown that

$$\Delta_p\left(\mathbb{E}\left(\sum_{i=1}^{n} \chi_{A_i} \otimes x_i \Big| \Sigma_1\right)\right) \leq \Delta_p\left(\sum_{i=1}^{n} \chi_{A_i} \otimes x_i\right).$$

The conditional expectation

$$\mathbb{E}(\,\cdot \mid \Sigma_1) : L^p(\mu, X) \to L^p(\mu, X),$$

is the continuous extension of the operator $\mathbb{E}(\,\cdot \mid \Sigma_1) \otimes id_X$ on $S_p(X)$ to $L^p(\mu, X)$; it is a linear contractive projection.

Chaney-Schaefer l-tensor product

Denote by $\|\cdot\|_l$ the norm on $E \otimes Y$ defined by

$$\|u\|_l = \inf\left\{\left\|\sum_{i=1}^{n} \|y_i\|\,|x_i|\right\| : u = \sum_{i=1}^{n} x_i \otimes y_i\right\}$$

- If $E = L^p$ for $1 \leq p < \infty$, then $\Delta_p = l$.

Conditional expectations on $E\widetilde{\otimes}_l Y$

The proof of the following result uses the fact that the l-norm is a left uniform, left injective (Riesz) cross norm.

Theorem

- *If (S_i) BL-filtration and and (T_i) a BS-filtration on E and Y respectively, then $(S_i \otimes_l T_i)$ is a BS-filtration on $E\widetilde{\otimes}_l F$.*
- *If (S_i) and (T_i) BL-filtrations on E and F respectively, then $(S_i \otimes_l T_i)$ is a BL-filtration on $E\widetilde{\otimes}_l F$.*

Lemma

If (S_i) *is a BL-filtration on* E *and* (T_i) *is a* BS*-filtration on* Y*, then* $\overline{\bigcup_{i=1}^{\infty} \mathcal{R}(S_i)} \ \widetilde{\otimes}_l \ \overline{\bigcup_{i=1}^{\infty} \mathcal{R}(T_i)} = \overline{\bigcup_{i=1}^{\infty} \mathcal{R}(S_i \otimes_l T_i)}$.

Theorem

If (S_i) *is a BL-filtration on the Banach lattice* E *and* (T_i) *is a* BS*-filtration on the Banach space* Y*, then*
$\mathcal{M}_{nc}\left(E \,\widetilde{\otimes}_l\, Y, S_i \otimes_l T_i\right) = \mathcal{M}_{nc}(E, S_i) \,\widetilde{\otimes}_l\, \mathcal{M}_{nc}(Y, T_i)$.

Description of convergent martingales

Theorem

Let (S_n) be a BL-filtration on a Banach lattice E and (T_n) is a BS-filtration on a Banach space Y. Then, in order for $M = (f_n, S_n \otimes_l T_n)_{n=1}^{\infty}$ to be a convergent martingale in $E\tilde{\otimes}_l Y$, it is necessary and sufficient that, for each $i \in \mathbb{N}$, there exist convergent martingales $\left(x_i^{(n)}, S_n\right)_{n=1}^{\infty}$ and $\left(y_i^{(n)}, T_n\right)_{n=1}^{\infty}$ in E and Y respectively such that, for each $n \in \mathbb{N}$, we have

$$f_n = \sum_{i=1}^{\infty} x_i^{(n)} \otimes y_i^{(n)},$$

where

$$\left\| \sum_{i=1}^{\infty} \left| \lim_{n\to\infty} x_i^{(n)} \right| \right\| < \infty \quad \textit{and} \quad \lim_{i\to\infty} \left\| \lim_{n\to\infty} y_i^{(n)} \right\| = 0.$$

Theorem

Let (Ω, Σ, μ) *denote a probability space,* $(\Sigma_n)_{n=1}^{\infty}$ *a filtration,* X *a Banach space and* $1 \leq p < \infty$. *Then, in order for* $(f_n, \Sigma_n)_{n=1}^{\infty}$ *to be a convergent martingale in* $L^p(\mu, X)$, *it is necessary and sufficient that, for each* $i \in \mathbb{N}$, *there exist a convergent martingale* $\left(x_i^{(n)}, \Sigma_n\right)_{n=1}^{\infty}$ *in* $L^p(\mu)$ *and* $y_i \in X$ *such that, for each* $n \in \mathbb{N}$, *we have*

$$f_n(s) = \sum_{i=1}^{\infty} x_i^{(n)}(s) y_i \quad \text{for all } s \in \Omega,$$

where

$$\left\| \sum_{i=1}^{\infty} \left| \lim_{n\to\infty} x_i^{(n)} \right| \right\|_{L^p(\mu)} < \infty \quad \text{and} \quad \lim_{i\to\infty} \|y_i\| \to 0.$$

Adjoints of conditional expectations

- It is easily verified that if (T_i) is a filtration on E, then the sequence of adjoint operators (T_i^*) is a filtration on E^*. However, it is not clear that (T_i^*) is a BL-filtration whenever (T_i) is a BL-filtration.

Definition

Let E be a Banach lattice with a non-empty quasi-interior Q_+. A filtration (T_i) on E is said to be quasi-interior preserving if $T_iQ_+ \subseteq Q_+$ for each $i \in \mathbb{N}$.

Lemma

Let E be a Banach lattice with a non-empty quasi-interior Q_+. If $T : E \to E$ is a positive projection, then $TQ_+ \subseteq Q_+$ if and only if there exists a quasi-interior point $0 < e \in E_+$ such that $Te = e$.

Theorem

Suppose that E is a Banach lattice possessing non-empty quasi-interior Q_+ and $T : E \to E$ is a bounded linear operator. Then $TQ_+ \subset Q_+$ if and only if $T^* : E^* \to E^*$ is strictly positive.

Corollary

Suppose that E is a Banach lattice possessing non-empty quasi-interior, then for any quasi-interior preserving filtration (T_i) on E, we have that (T_i^*) is a BL-filtration on E^*.

A linear map $T : E \to Y$ is called cone absolutely summing if for every positive summable sequence (x_n) in E, the sequence (Tx_n) is absolutely summable in Y.

The space

$$\mathcal{L}^{\text{cas}}(E, Y) = \{T : E \to Y : T \text{ is cone absolutely summing }\}$$

is a Banach space with respect to the norm defined by

$$\|T\|_{\text{cas}} = \sup \left\{ \sum_{i=1}^{n} \|Tx_i\| : x_1, \ldots, x_n \in E_+, \left\| \sum_{i=1}^{n} x_i \right\| = 1, n \in \mathbb{N} \right\}$$

for all $T \in \mathcal{L}^{\text{cas}}(E, Y)$.
There is a canonical embedding $E \otimes Y$ into $\mathcal{L}^{\text{cas}}(E^*, Y)$, given by

$$\sum_{i=1}^{n} x_i \otimes y_i = u \mapsto L_u$$

where $L_u x^* = \sum_{i=1}^{n} \langle x_i, x^* \rangle y_i$, and $\| \cdot \|_l = \| \cdot \|_{\text{cas}}$.

Martingales of cas maps

Theorem

Suppose that (T_i) *is a BL-filtration on* E*. Then the sequence* $(\widehat{T}_i)$ *of maps* $\widehat{T}_i : \mathcal{L}^{\mathrm{cas}}(E,Y) \to \mathcal{L}^{\mathrm{cas}}(E,Y)$*, defined by* $\widehat{T}_i F = F \circ T_i$ *for each* $F \in \mathcal{L}^{\mathrm{cas}}(E,Y)$ *and* $i \in \mathbb{N}$*, is a sequence of contractive projections on* $\mathcal{L}^{\mathrm{cas}}(E,Y)$ *with* $\widehat{T}_{i\wedge j} = \widehat{T}_i \widehat{T}_j$*.*

Definition

Suppose that (T_i) is a BL-filtration on E. Then $(\widehat{T}_i)$, as defined above, is called the filtration on $\mathcal{L}^{\mathrm{cas}}(E,Y)$ induced by (T_i).

The key to characterizations of the Radon Nikodým property

Definition

If (f_i, T_i) is a martingale in Y, then (f_i, T_i) is called fixed if there exists $f \in Y$ such that $f_i = T_i f$ for all $i \in \mathbb{N}$. In this case, (f_i, T_i) is said to be fixed on f. Let $\mathcal{M}_{\mathrm{f}}(Y, T_i) = \{(f_i, T_i) \in \mathcal{M}(Y, T_i) : \exists f \in Y \text{ so that } T_i f = f_i \; \forall \; i \in \mathbb{N}\}$.

Theorem

Let E be a Banach lattice with order continuous dual and Y a Banach space. Suppose that (T_i) is a BL-filtration on E and $(\widehat{T}_i)$ is the filtration on $\mathcal{L}^{\mathrm{cas}}(E, Y)$ induced by (T_i). Then $\mathcal{M}_{\mathrm{f}}(\mathcal{L}^{\mathrm{cas}}(E, Y), \widehat{T}_i) = \mathcal{M}(\mathcal{L}^{\mathrm{cas}}(E, Y), \widehat{T}_i)$.

Theorem

Let Y be a Banach space. Then the following statements are equivalent:

(a) *Y has the Radon Nikodým property.*

(b) *$E^*\widetilde{\otimes}_l Y = \mathcal{L}^{\mathrm{cas}}(E, Y)$ for all separable Banach lattices E with order continuous dual.*

(c) *$\mathcal{M}(E^*\widetilde{\otimes}_l Y, T_i^* \otimes_l \mathrm{id}_Y) = \mathcal{M}_{\mathrm{f}}(E^*\widetilde{\otimes}_l Y, T_i^* \otimes_l \mathrm{id}_Y)$ for all separable Banach lattices E with order continuous dual and all BL-filtrations (T_i) on E.*

(d) *$\mathcal{M}(E\widetilde{\otimes}_l Y, T_i \otimes_l \mathrm{id}_Y) = \mathcal{M}_{\mathrm{nc}}(E\widetilde{\otimes}_l Y, T_i \otimes_l \mathrm{id}_Y)$ for all separable reflexive Banach lattices E and all complemented, quasi-interior preserving BL-filtrations (T_i) on E.*

(e) *$\mathcal{M}(E\widetilde{\otimes}_l Y, T_i \otimes_l \mathrm{id}_Y) = \mathcal{M}(E, T_i)\widetilde{\otimes}_l Y$ for all separable reflexive Banach lattices E and all complemented, quasi-interior preserving BL-filtrations (T_i) on E.*

Corollary

Let Y be a Banach space. Then Y is an Asplund space (i.e. Y^* has the Radon-Nikodým property) if and only if $E^* \widetilde{\otimes}_l Y^* = (E \widetilde{\otimes}_l Y)^*$ for all separable Banach lattices E with order continuous dual.

Corollary

Let Y be a Banach space. Then the following conditions are equivalent:

(a) *Y has the Radon Nikodým property.*

(b) *For every separable reflexive Banach lattice E and every complemented, quasi-interior preserving BL-filtration (T_i) on E, we have $(f_n) \in \mathcal{M}(E\tilde{\otimes}_l Y, T_i \otimes_l \mathrm{id}_Y)$ if and only if for each $i \in \mathbb{N}$, there exist $(x_i^{(n)}, T_n)_{n=1}^{\infty} \in \mathcal{M}_{\mathrm{nc}}(E, T_i)$ and $y_i \in Y$ such that, for each $n \in \mathbb{N}$, we have $f_n = \sum_{i=1}^{\infty} x_i^{(n)} \otimes y_i$, where $\| \sum_{i=1}^{\infty} | \lim_{n\to\infty} x_i^{(n)} | \| < \infty$ and $\lim_{i\to\infty} \|y_i\| = 0$.*

Discrete time stochastic processes

- Gao N., Xanthos F., Unbounded order convergence and applications to martingales without probability, preprint, (2013)
- Kuo, W.-C., Stochastic Processes on Vector Lattices. Thesis, University of the Witwatersrand (2006)
- Kuo, W.-C., Labuschagne, C.C.A., Watson, B.A.: Discrete time stochastic processes on Riesz spaces. Indag. Math. 15, 435 – 451 (2004)
- Kuo, W.-C., Labuschagne, C.C.A., Watson, B.A.: An upcrossing theorem for martingales on Riesz spaces, Soft methodology and random information systems, pp. 101 – 108. Springer, Berlin (2004)
- Kuo, W.-C., Labuschagne, C.C.A., Watson, B.A.: Conditional expectation on Riesz spaces. J. Math. Anal. Appl. 303, 509 – 521 (2005)

- Kuo, W.-C., Labuschagne, C.C.A., Watson, B.A.: Zero-one laws for Riesz space and fuzzy random variables, Fuzzy logic, soft computing and computational intelligence, pp. 393 – 397. Springer and Tsinghua University Press, Beijing, China (2005)
- Kuo, W.-C., Labuschagne, C.C.A., Watson, B.A.: Convergence of Riesz space martingales. Indag. Math. 17, 271 – 283 (2006)
- Kuo,W.-C., Labuschagne, C.C.A., Watson, B.A.: Ergodic theory and the strong law of large numbers on Riesz spaces. J. Math. Anal. Appl. 325 (2007), 422 – 437.

- KUO, WEN-CHI; LABUSCHAGNE, COENRAAD C. A.; WATSON, BRUCE A., Amarts on Riesz spaces. Acta Math. Sin. (Engl. Ser.) 24 (2008), 329 – 342.
- KUO, WEN-CHI; VARDY, JESSICA JOY; WATSON, BRUCE ALASTAIR, Mixingales on Riesz spaces. J. Math. Anal. Appl. 402 (2013), no. 2, 731 – 738.
- M. Korostenski, C.C.A. Labuschagne, B.A. Watson, Reverse martingales in Riesz spaces, Proceedings of the 18th International Workshop on Operator Theory and Applications (IWOTA 2007): Operator Theory: Advances and Applications, 195, 213–230 (2009).

- C.C.A. Labuschagne, B.A. Watson, Discrete time stochastic integrals in Riesz spaces, Positivity 14 (2010), 859 – 875.
- WATSON, BRUCE A., An Andô-Douglas type theorem in Riesz spaces with a conditional expectation. Positivity 13 (2009), no. 3, 543 – 558.

- Grobler J.J., Continuous stochastic processes in Riesz spaces: the Doob-Meyer decomposition, Positivity 14 (2010), 731 – 751
- GROBLER, J. J., Doob's optional sampling theorem in Riesz spaces. Positivity 15 (2011), 617 – 637.
- GROBLER, J.J., Jensen's and martingale inequalities in Riesz spaces. Indagationes Mathematica, in press (2013).
- VARDY, JESSICA J.; WATSON, BRUCE A. Erratum to: Markov processes on Riesz spaces [MR2929096]. Positivity 16 (2012), no. 2, 393.
- VARDY, JESSICA J.; WATSON, BRUCE A., Markov processes on Riesz spaces. Positivity 16 (2012), no. 2, 373 – 391.

Banach Space Applications

- S.F. Cullender, C.C.A. Labuschagne, A description of norm-convergent martingales on vector valued L^p-spaces, Journal of Mathematical Analysis and Applications 323 (2006), 119–130.
- S.F. Cullender, C.C.A. Labuschagne, Unconditional Schauder decompositions and stopping times in the Lebesgue-Bochner spaces, Journal of Mathematical Analysis and Applications 336 (2007), 849–864.
- S.F. Cullender, C.C.A. Labuschagne, Convergent martingales of operators and the Radon-Nikodým property in Banach spaces, Proceedings of the American Mathematical Society 136 (2008), 3883–3893.
- GESSESSE, H., TROITSKY, V., Martingales in Banach lattices, II. Positivity 15 (2011), no. 1, 49–55.

- C.C.A. Labuschagne, A Banach lattice approach to convergent integrably bounded set-valued martingales and their positive parts, Journal of Mathematical Analysis and Applications 342 (2008), 780–797.
- C.C.A. LABUSCHAGNE, A.L. PINCHUCK, Doob's decomposition of set-valued submartingales via ordered near vector spaces, Quaestiones Mathematicae, 32 (2009), 247–264.
- C.C.A. LABUSCHAGNE, V. MARRAFFA, Operator martingale decompositions and the Radon-Nikodým property in Banach spaces, Journal of Mathematical Analysis and Applications 363 (2010), 357–365.
- Troitsky, V., Martingales in Banach lattices. Positivity 9, 437 – 456 (2005)

THANK YOU FOR YOUR ATTENTION