Approximation of solutions to some abstract Cauchy problems by means of Szász-Mirakjan-Kantorovich operators

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joint work with
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Statement of the problem

Let A be a closed operator defined on a suitable domain $D(A)$ of a certain Banach space $(E, \|\cdot\|)$.
It is well-known that if $(A, D(A))$ generates a C_0-semigroup $(T(t))_{t\geq 0}$ on E, then the abstract Cauchy problem

$$\begin{cases} \dfrac{du}{dt}(t) = Au(t), & t \geq 0, \\ u(0) = u_0, & u_0 \in D(A) \end{cases} \tag{ACP}$$

associated with $(A, D(A))$, admits a unique solution $u : [0, +\infty[\to E$ given by [1]

$$u(t) = T(t)(u_0) \quad \text{for every } t \geq 0 \tag{1}$$

[1]See, e.g., Chapter A-II in [R. Nagel (Ed.), One-parameter semigroups of positive operators, Lecture Notes in Math. **1184**, Springer-Verlag (Berlin, 1986)].

Statement of the problem

Our general aim is to investigate the possibility to determine a suitable sequence $(L_n)_{n\geq 1}$ of positive linear operators on E such that for every $t \geq 0$ and for every sequence $(\rho_n)_{n\geq 1}$ of positive integers such that $\lim_{n\to\infty} \frac{\rho_n}{n} = t$, one has

$$T(t)(f) = \lim_{n\to\infty} L_n^{\rho_n}(f) \qquad (f \in E), \tag{2}$$

where each $L_n^{\rho_n}$ denotes the iterate of L_n of order ρ_n.

This technique, based on approximation theory, was developed by F. Altomare in the nineties [2] and allows to obtain some qualitative properties of the semigroup, and hence of the solution to (ACP), by means of similar ones held by the operators L_n thanks to the representation formula (2).

[2] [F. Altomare, *Approximation theory methods for the study of diffusion equations*, Approximation Theory, Proc. IDOMAT, 75 M-W- Müler, M. Felten, D.H. Mache Eds., Math. Res., 86, Akademie Verlag, Berlin, 1995, 9-26.

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A consequence of Trotter's theorem

Proposition

Let $(A, D(A))$ be the generator of a C_0-semigroup $(T(t))_{t\geq 0}$ on E for which there exist $M \geq 1$ and $\omega \in \mathbf{R}$ such that $\|T(t)\| \leq Me^{\omega t}$ $(t \geq 0)$. If D is a core for $(A, D(A))$ (i.e., a linear subspace of $D(A)$ dense in $D(A)$ for the graph norm $\|u\|_A := \|u\| + \|Au\|$ $(u \in D(A))$) and $(L_n)_{n\geq 1}$ is a sequence of bounded linear operators on E such that

(i) $\|L_n^k\| \leq Me^{\omega k/n}$ *for every* $n, k \geq 1$;

(ii) $\lim_{n\to\infty} n(L_n(u) - u) = A(u)$ *for every* $u \in D$,

then, for every $u \in E$,

$$T(t)(u) = \lim_{n\to\infty} L_n^{\rho_n}(u)$$

where $t \geq 0$ *and* $(\rho_n)_{n\geq 1}$ *is a sequence of positive integers such that* $\lim_{n\to\infty} \frac{\rho_n}{n} = t$.

(See Thm 2.1 in [F. Altomare, V. L., S. Milella, *Cores for second-order differential operators on real intervals*, Commun. Appl. Anal. **13** (2009), no. 4, 477-496.])

Notation

$C([0,+\infty[) := \{f : [0,+\infty[\to \mathbf{R} \mid f \text{ continuous on } [0,+\infty[\}$

$C^2([0,+\infty[) := \{f \in C([0,+\infty[) \mid f \text{ twice continuously derivable on } [0,+\infty[\}$

$C_b([0,+\infty[) := \{f \in C([0,+\infty[) \mid f \text{ bounded}\}$

$(C_b([0,+\infty[), \|\cdot\|_\infty, \leq)$ Banach lattice

$$C_*([0,+\infty[) := \left\{ f \in C_b([0,+\infty[) \mid \text{ there exists } \lim_{x\to+\infty} f(x) \in \mathbf{R} \right\}$$

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Banach sublattices of $C_b([0,+\infty[)$

$K([0,+\infty[) := \{f \in C([0,+\infty[) \mid \mathrm{Supp}(f) \text{ is compact}\}$

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Moreover, for every $m \geq 1$, set $w_m(x) := (1 + x^m)^{-1}$ $(x \geq 0)$ and

$$E_m := \left\{ f \in C([0, +\infty[) \mid \sup_{x \geq 0} w_m(x)|f(x)| \in \mathbf{R} \right\}$$

$(E_m, \| \cdot \|_m, \leq)$ Banach lattice, where $\|f\|_m := \sup_{x \geq 0} w_m(x)|f(x)|$ $(f \in E_m)$

$$E_m^* := \left\{ f \in E_m \mid \lim_{x \to +\infty} w_m(x) f(x) \in \mathbf{R} \right\}$$

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Banach sublattices of E_m

The class of second-order differential operators

For a fixed $0 \le l \le 2$, let V_l be the second-order differential operator defined by

$$V_l(u)(x) := xu''(x) + \frac{l}{2}u'(x) \quad (x > 0, u \in C^2(]0, +\infty[)). \tag{3}$$

Note that, up to a change of variable, the operators in (3) are strictly connected with a backward equation of the type

$$\frac{\partial u}{\partial t}(x,t) = ax^{2-p}\frac{\partial^2 u}{\partial x^2}(x,t) + bx^{1-p}\frac{\partial u}{\partial x}(x,t), \quad x, t > 0, \tag{4}$$

with $p > 0$ and $b > (1-p)a$, that corresponds, for example, to the radial component of the N-dimensional Brownian motion ($N \ge 1$) or to a stochastic process that is the limit of a sequence of random walks.

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It is known that such an operator, defined on a suitable domain of continuous functions on $[0, +\infty[$, as well as on weighted continuous functions on $[0, +\infty[$, generates a strongly continuous semigroup [3].

Moreover, it generates, in a suitable domain, a Feller semigroup also in $L^p([0, +\infty[)$. [4]

[3] F. Altomare, S. Milella, *Degenerate differential equations and modified Szász-Mirakjan operators*, Rend. Circ. Mat. Palermo, **59** (2010), 227-250.

[4] S. Fornaro, G. Metafune, D. Pallara, J. Prüss, *L^p-theory for some elliptic and parabolic problems with first order degeneracy at the boundary*, J. Math. Pures Appl., **87** (2007), 367-393.

Let

$$S([0,+\infty[) := \{f : [0,+\infty[\to \mathbf{R} \mid \text{ there exist } M \geq 0 \text{ and } \alpha \in \mathbf{R} \text{ such that } |f(x)| \leq Me^{\alpha x}\}$$

In 1941 G.M. Mirakjan (see also, e.g., [J. Favard, 1944], [O. Szász, 1950]) introduced the Szász-Mirakjan operators

$$S_n(f)(x) := e^{-nx} \sum_{k=0}^{\infty} \frac{(nx)^k}{k!} f\left(\frac{k}{n}\right) \quad (n \geq 1, x \geq 0) \tag{5}$$

defined for every function $f \in S([0,+\infty[)$.

Szász-Mirakjan-Kantorovich operators

Let $T([0,+\infty[)$ be the space of all functions $f \in L^1_{loc}([0,+\infty[)$ such that $F \in S([0,+\infty[)$, where

$$F(x) := \int_0^x f(t)dt \quad (x \geq 0).$$

In 1954 P.L. Butzer considered the so-called Szász-Mirakjan-Kantorovich operators defined by setting, for every $n \geq 1$, $f \in T([0,+\infty[)$ and $x \geq 0$,

$$K_n(f)(x) := e^{-nx} \sum_{k=0}^{\infty} \frac{(nx)^k}{k!} \left[n \int_{\frac{k}{n}}^{\frac{k+1}{n}} f(t)\, dt \right]. \tag{6}$$

Modified Szász-Mirakjan-Kantorovich operators

Following an idea in

- F. Altomare, V. L., *On a sequence of positive linear operators associated with a continuous selection of Borel measures*, Mediterr. j. math. 3 (2006), 363-382,

we modify the K_n's as follows.

Let $(a_n)_{\geq 1}$ and $(b_n)_{n\geq 1}$ be two sequencse in $[0,1]$ such that $a_n < b_n$ for every $n \geq 1$. Then, for every $n \geq 1$, $f \in T([0,+\infty])$ and $x \geq 0$,

$$M_n(f)(x) := e^{-nx} \sum_{k=0}^{\infty} \frac{(nx)^k}{k!} \left[\frac{n}{b_n - a_n} \int_{\frac{k+a_n}{n}}^{\frac{k+b_n}{n}} f(t)\, dt \right]. \qquad (7)$$

Note that

$$\bigcup_{k=0}^{\infty} \left[\frac{k+a_n}{n}, \frac{k+b_n}{n} \right] \subsetneq [0,+\infty[$$

Moreover, if $a_n = 0$ and $b_n = 1$ $(n \geq 1)$, then $M_n = K_n$.

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Observe that

$$S([0,+\infty[) \cap C([0,+\infty[) \subset T([0,+\infty[)$$

and, if $m \geq 1$ and $p \in [1,+\infty[$,

$$E_m \subset T([0,+\infty[) \quad \text{and} \quad L^p([0,+\infty[) \subset T([0,+\infty[).$$

About M_n's operators

- F. Altomare, M. Cappelletti Montano and V. L., *On a modification of Szász-Mirakjan-Kantorovich operators*, Results. Math. Vol. **63**, Issue 3 (2013), 837-863, DOI: 10.1007/s0025-012-0236-z.

- Approximation properties of $(M_n)_{n\geq 1}$
  - on $C_b([0,+\infty[)$, $C_*([0,+\infty[)$, $C_0([0,+\infty[)$
  - on E_m, E_m^*, E_m^0
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- Estimates of the rate of convergence

- M. Cappelletti Montano and V. L., *Approximation of some Feller semigroups associated with a modification of Szász-Mirakjan-Kantorovich operators*, Acta Math. Hunga., **139**, Issue 3 (2013), 255-275, DOI: 10.1007/s10474-012-0267-7.

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1. $M_n(\mathbb{P}_m) \subset \mathbb{P}_m$ for every $n \geq 1$, where $\mathbb{P}_m$ is the space of (the restriction to $[0, +\infty[$ of) all polynomials of degree not greater than m, $m \geq 1$.
2. Fix $f \in C_b([0, +\infty[)$, then
$$f \text{ is increasing} \iff M_n(f) \text{ is increasing for every } n \geq 1.$$
$$f \text{ is convex} \iff M_n(f) \text{ is convex for every } n \geq 1.$$
3. For every $n \geq 1$, $M \geq 0$ and $\alpha \in]0, 1]$ one has $M_n(\mathrm{Lip}_M \alpha) \subset \mathrm{Lip}_M \alpha$, where $\mathrm{Lip}_M \alpha$ is the class of all continuous functions on $[0, +\infty[$ that are Lipschitz continuous of order α (with Lipschitz constant M) on $[0, +\infty[$.
4. If $f \in C_b([0, +\infty[)$ is convex and increasing (resp. convex and decreasing), then for every $n \geq 1$
$$f \leq M_n(f) \quad \text{on } [0, +\infty[\qquad (\text{resp. } M_n(f) \leq f \text{ on } [0, +\infty[)$$
Moreover, if $f \in C_b([0, +\infty[)$ is convex and increasing and there exist $a, b \in \mathbf{R}$, $0 \leq a < b \leq 1$ such that $a_n = a$ and $b_n = b$ for every $n \geq 1$, then $M_{n+1}(f) \leq M_n(f)$ for every $n \geq 1$ on $[0, +\infty[$.

Qualitative properties of the M_n's

1. $M_n(\mathbb{P}_m) \subset \mathbb{P}_m$ for every $n \geq 1$, where $\mathbb{P}_m$ is the space of (the restriction to $[0, +\infty[$ of) all polynomials of degree not greater than m, $m \geq 1$.
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3. For every $n \geq 1$, $M \geq 0$ and $\alpha \in]0, 1]$ one has $M_n(\mathrm{Lip}_M \alpha) \subset \mathrm{Lip}_M \alpha$, where $\mathrm{Lip}_M \alpha$ is the class of all continuous functions on $[0, +\infty[$ that are Lipschitz continuous of order α (with Lipschitz constant M) on $[0, +\infty[$.
4. If $f \in C_b([0, +\infty[)$ is convex and increasing (resp. convex and decreasing), then for every $n \geq 1$
$$f \leq M_n(f) \quad \text{on } [0, +\infty[\qquad (\text{resp. } M_n(f) \leq f \text{ on } [0, +\infty[)$$
Moreover, if $f \in C_b([0, +\infty[)$ is convex and increasing and there exist $a, b \in \mathbf{R}$, $0 \leq a < b \leq 1$ such that $a_n = a$ and $b_n = b$ for every $n \geq 1$, then $M_{n+1}(f) \leq M_n(f)$ for every $n \geq 1$ on $[0, +\infty[$.

Asymptotic formulas

Let $(M_n)_{n\geq 1}$ be the sequence of operators given by

$$M_n(f)(x) := e^{-nx}\sum_{k=0}^{\infty}\frac{(nx)^k}{k!}\left[\frac{n}{b_n-a_n}\int_{\frac{k+a_n}{n}}^{\frac{k+b_n}{n}} f(t)\,dt\right]$$

and from now on suppose that there exists

$$l := \lim_{n\to\infty}(a_n+b_n)\in \mathbf{R}.$$

Clearly $0\leq l\leq 2$. Then, considering

$$V_l(u)(x) := xu''(x)+\frac{l}{2}u'(x) \quad (x>0, u\in C^2(]0,+\infty[)),$$

we get

Theorem

Set $m \geq 2$. Then for every $u \in C^2([0,+\infty[) \cap E_m^0$ such that u'' is uniformly continuous and bounded,

$$\lim_{n\to\infty} n(M_n(u) - u) = V_l(u) \quad \text{in } E_m^0. \tag{8}$$

In particular,

$$\lim_{n\to\infty} n(M_n(u) - u) = V_l(u) \quad \text{uniformly on compact subsets of } [0,+\infty[.$$

Further, for every $u \in K^2([0,+\infty[)$,

$$\lim_{n\to\infty} n(M_n(u) - u) = V_l(u) \quad \text{uniformly on } [0,+\infty[. \tag{9}$$

Boundary conditions

Consider two extensions of V_l associated with the following classical boundary conditions:

$$\lim_{x\to 0^+} V_l(u)(x) \in \mathbf{R} \quad \text{and} \quad \lim_{x\to +\infty} V_l(u)(x) = 0 \quad \text{if } l = 2 \qquad (10)$$

or

$$\lim_{x\to 0^+} V_l(u)(x) = 0 \quad \text{and} \quad \lim_{x\to +\infty} V_l(u)(x) = 0 \quad \text{if } l < 2 \qquad (11)$$

$(V_l, D_0(V_l))$ where

$$D_0(V_l) := \{u \in C_0([0,+\infty[) \cap C^2(]0,+\infty[) \mid u \text{ satisfies (10) or (11)}\}$$

$(V_l, D_*(V_l))$ where

$$D_*(V_l) := \{u \in C_*([0,+\infty[) \cap C^2(]0,+\infty[) \mid u \text{ satisfies (10) or (11)}\}.$$

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Semigroups generated by V_l

Proposition

The operators $(V_l, D_0(V_l))$ and $(V_l, D_*(V_l))$ generate Feller semigroups $(T_0(t))_{t\geq 0}$ on $C_0([0,+\infty[)$ and $(T_*(t))_{t\geq 0}$ on $C_*([0,+\infty[)$, respectively. Moreover, set $D_1 := \{u \in K^2([0,+\infty[) \,|\, \lim_{x\to 0^+} u'(x) = 0\}$ and $S := \{u \in C^2(]0,+\infty[) \,|\, u$ is constant on a neighborhood of $+\infty\}$. Then

1. *if $l < 2$ the space D_1 is a core for $(V_l, D_0(V_l))$ and the space generated by $D_1 \cup S$ is a core for $(V_l, D_*(V_l))$;*
2. *if $l = 2$ the space $K^2([0,+\infty[)$ is a core for $(V_l, D_0(V_l))$ and the space generated by $K^2([0,+\infty[) \cup S$ is a core for $(V_l, D_*(V_l))$.*

Proof. It is sufficient to apply Theorems 1, 3 and 4 in [F. Altomare, S. Milella, *Degenerate differential equations and modified Szász-Mirakjan operators*, Rend. Circ. Mat. Palermo, **59** (2010), 227-250].

Boundary conditions

For every $u \in E_m^0 \cap C^2(]0, +\infty[)$ such that

$$\lim_{x \to 0^+} V_l(u)(x) = 0 \text{ and } \lim_{x \to +\infty} w_m(x) V_l(u)(x) = 0 \tag{12}$$

$V_l(u)$ can be continuously extended on $[0, +\infty[$ and its extension is on E_m^0. Let $(W_l, D_m(W_l))$ such that

$$W_l(u)(x) = \begin{cases} V_l(u)(x) & \text{if } x > 0, \\ 0 & \text{if } x = 0, \end{cases} \tag{13}$$

for every u belonging to

$$D_m(W_l) := \{u \in E_m^0 \cap C^2(]0, +\infty[) \,|\, u \text{ satisfies } (12)\}$$

Semigroups generated by V_l

Proposition

The operator $(W_l, D_m(W_l))$ is the generator of a strongly continuous semigroup $(T_m(t))_{t\geq 0}$ on E_m^0 such that $\|T(t)\|_{E_m^0} \leq e^{\omega_m t}$ for each $t \geq 0$, ω_m being

$$\omega_m := \sup_{x>0} \frac{2(m^2 - m)x^m + mlx^{m-1}}{2(1+x^m)}. \tag{14}$$

Moreover the set $D_1 := \{u \in K^2([0,+\infty[) \mid \lim_{x\to 0^+} u'(x) = 0\}$ is a core for $(W_l, D_m(W_l))$.

Proof. It is sufficient to apply Theorems 2-4 in [F. Altomare, S. Milella, *Degenerate differential equations and modified Szász-Mirakjan operators*, Rend. Circ. Mat. Palermo, 59 (2010), 227-250].

Approximation of the semigroup

Let $(T_m(t))_{m\geq 1}$ be the C_0-semigroup generated by $(W_I, D_m(W_I))$.
By means of $\|M_n\|_{E_m^0} \leq 1 + d_m/n$, for every $k, n \geq 1$,

$$\|M_n^k\|_{E_m^0} \leq \left(1 + \frac{d_m}{n}\right)^k \leq e^{d_m \frac{k}{n}} \leq e^{\max\{d_m, \omega_m\}\frac{k}{n}}.$$

Moreover, $D_1 := \{u \in K^2([0,+\infty[) \mid \lim_{x\to 0^+} u'(x) = 0\}$ is a core for $(W_I, D_m(W_I))$ and, for every $u \in D_1 \subset K^2([0,+\infty[)$,

$$\lim_{n\to\infty} n(M_n(u) - u) = V_I(u) \quad \text{uniformly on} \quad [0,+\infty[,$$

and accordingly, in E_m^0 (since $\|\cdot\|_w \leq \|w\|_\infty \|\cdot\|_\infty$).
On account of the consequence of Trotter's theorem we get that, if $t \geq 0$ and $(\rho_n)_{n\geq 1}$ is a sequence of positive integers such that $\lim_{n\to\infty} \rho_n/n = t$, for every $f \in E_m^0$ we get

$$\lim_{n\to\infty} M_n^{\rho_n}(f) = T_m(t) \quad \text{in } E_m^0,$$

and hence uniformly on compact subsets of $[0,+\infty[$.

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Moreover, $D_1 := \{u \in K^2([0, +\infty[) \mid \lim_{x\to 0^+} u'(x) = 0\}$ is a core for $(W_I, D_m(W_I))$ and, for every $u \in D_1 \subset K^2([0, +\infty[)$,

$$\lim_{n\to\infty} n(M_n(u) - u) = V_I(u) \quad \text{uniformly on} \quad [0, +\infty[,$$

and accordingly, in E_m^0 (since $\|\cdot\|_w \leq \|w\|_\infty \|\cdot\|_\infty$).
On account of the consequence of Trotter's theorem we get then, if $t \geq 0$ and $(\rho_n)_{n\geq 1}$ is a sequence of positive integers such that $\lim_{n\to\infty} \rho_n/n = t$, for every $f \in E_m^0$ we get

$$\lim_{n\to\infty} M_n^{\rho_n}(f) = T_m(t) \quad \text{in } E_m^0$$

and hence uniformly on compact subsets of $[0, +\infty[$.

Approximation of the semigroup

Theorem

Let $(T_0(t))_{t\geq 0}$ *(resp.,* $(T_*(t))_{t\geq 0}$*) be the Feller semigroup generated by the operators* $(V_I, D_0(V_I))$ *(resp.,* $(V_I, D_*(V_I))$*). Then, if* $t \geq 0$ *and* $(\rho_n)_{n\geq 1}$ *is a sequence of positive integers such that* $\lim_{n\to\infty} \rho_n/n = t$*, for every* $f \in C_0([0,+\infty[)$

$$\lim_{n\to\infty} M_n^{\rho_n}(f) = T_0(t) \quad \textit{uniformly on } [0,+\infty[$$

(resp., for every $f \in C_*([0,+\infty[)$

$$\lim_{n\to\infty} M_n^{\rho_n}(f) = T_*(t) \quad \textit{uniformly on } [0,+\infty[)$$

Approximation of the solution of differential problems

Denote by $(A, D(A))$ one of the following operators considered above:

$$(W_l, D_m(W_l)), \quad (V_l, D_0(V_l)) \quad \text{or} \quad (V_l, D_*(V_l))$$

and by $(T(t))_{t\geq 0}$ the corresponding semigroup.
Consider the differential problem

$$\begin{cases} \dfrac{\partial u}{\partial t}(x,t) = A(u(\cdot,t)), & x \geq 0, t \geq 0, \\ u(x,0) = u_0(x), & u_0 \in D(A), x \geq 0 \end{cases} \tag{15}$$

associated with $(A, D(A))$.
Then

$$u(x,t) = T(t)(u_0)(x) = \lim_{n\to\infty} M_n^{[nt]}(u_0)(x) \tag{16}$$

where $\rho_n = [nt]$ is the integer part of nt.

Qualitative properties of the solution

- u_0 positive $\Longrightarrow u(\cdot, t)$ positive for every $t \geq 0$
- $u_0 \in \mathbb{P}_m \Longrightarrow u(\cdot, t) \in \mathbb{P}_{[nt]m}$
- u_0 increasing $\Longrightarrow u(\cdot, t)$ increasing for every $t \geq 0$
- u_0 convex $\Longrightarrow u(\cdot, t)$ convex for every $t \geq 0$
- $u_0 \in \mathrm{Lip}_M \alpha \Longrightarrow u(\cdot, t) \in \mathrm{Lip}_M \alpha$ for every $t \geq 0$
- u_0 convex and increasing $\Longrightarrow u_0 \leq u(\cdot, t)$ for every $t \geq 0$

The L^p-case

Denote by $e_1(t) = t$ $(t \geq 0)$ and set

$$D_p^0 = \{u \in L^p(]0,+\infty[) \cap W^{2,p}_{loc}(]0,+\infty[) \mid u', \sqrt{e_1}\,u', e_1 u'' \in L^p(]0,+\infty[) \text{ and } \lim_{x\to 0^+} u(x) = 0\}.$$

Then [5]

Theorem

If $1 < p < 2$, the operator (V_l, D_p^0) generates a Feller semigroup $(T_p(t))_{t\geq 0}$ in $L^p(]0,+\infty[)$.
Moreover $D := \{u \in K^\infty(\mathbf{R}) \mid u(0) = 0\} \subset K^2([0,+\infty[)$ is a core for (V_l, D_p^0), where $K^\infty(\mathbf{R})$ is the space of all continuous real valued functions with compact support that are infinitely many times derivable on $\mathbf{R}$.

[5] S. Fornaro, G. Metafune, D. Pallara, J. Prüss, *L^p-theory for some elliptic and parabolic problems with first order degeneracy at the boundary*, J. Math. Pures Appl., 87 (2007), 367-393.

The L^p-case

Theorem

Fix $1 \le p < +\infty$. *Then,* $M_n(L^p([0,+\infty[)) \subset L^p([0,+\infty[)$ *and*

$$\|M_n\|_{L^p,L^p} \le (b_n - a_n)^{-1/p}.$$

Moreover, if $(1/(b_n - a_n))_{n\ge 1}$ *is bounded, then for every* $f \in L^p([0,+\infty[)$

$$\lim_{n\to\infty} M_n(f) = f \quad \text{in } L^p([0,+\infty[).$$

Theorem

Fix $1 \le p < +\infty$ *and assume that* $l := \lim_{n\to\infty}(a_n + b_n) \in \mathbf{R}$.
Then, for every $v \in K^2([0,+\infty[)$,

$$\lim_{n\to\infty} n(M_n(v) - v) = V_l(v) \quad \text{in } L^p([0,+\infty[).$$

The L^p-case

Proposition

Assume that either

(a) $a_n = 0$ *and* $b_n = 1$ *for every* $n \geq 1$

or

(b) *the following properties hold true:*

(i) $0 < b_n - a_n < 1$ *for every* $n \geq 1$;

(ii) *there exist* $\lim_{n\to\infty} a_n = 0$ *and* $\lim_{n\to\infty} b_n = 1$;

(iii) $M_1 := \sup_{n\geq 1} n(1 - b_n + a_n) < +\infty$.

Then, for every $p \geq 1$ *there exists* $\tau_p \geq 0$*, depending on the sequences* $(a_n)_{n\geq 1}$ *and* $(b_n)_{n\geq 1}$*, such that for every* $k \geq 1$ *and* $n \geq 1$*,*

$$\|M_n^k\|_{L^p,L^p} \leq e^{\frac{k}{n}\tau_p}. \tag{17}$$

For example, fixed $\alpha \geq 1$,

$$a_n := \frac{1}{2}\left(1 + \frac{1}{2n^\alpha} - \frac{n\alpha}{n^\alpha + 1}\right) \quad \text{and} \quad b_n := \frac{1}{2}\left(1 + \frac{1}{2n^\alpha} + \frac{n\alpha}{n^\alpha + 1}\right).$$

The L^p-case

Proposition

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Theorem

If $t \geq 0$ *and* $(\rho_n)_{n\geq 1}$ *is a sequence of positive integers such that* $\lim_{n\to\infty} \rho_n/n = t$, *for every* $f \in L_p(]0,+\infty[)$, $1 < p < 2$,

$$\lim_{n\to\infty} M_n^{\rho_n}(f) = T_p(t) \quad \text{in} \;\; L^p(]0,+\infty[).$$

References

F. Altomare, M. Cappelletti Montano and V. L., *On a modification of Szász-Mirakjan-Kantorovich operators*, Results. Math. Vol. **63**, Issue 3 (2013), 837-863, DOI: 10.1007/s0025-012-0236-z.

M. Cappelletti Montano and V. L., *Approximation of some Feller semigroups associated with a modification of Szász-Mirakjan-Kantorovich operators*, Acta Math. Hunga., **139**, Issue 3 (2013), 255-275, DOI: 10.1007/s10474-012-0267-7.

THANKS FOR YOUR ATTENTION

Approximation results (on continuous function spaces)

Theorem

The following statements hold true:

(a) M_n *is a positive continuous linear operator from* $C_b([0,+\infty[)$ *into itself and* $\|M_n\|_{C_b([0,+\infty[)} = 1$.

(b) $M_n(C_0([0,+\infty[)) \subset C_0([0,+\infty[)$ *for every* $n \geq 1$.

Remark

$M_n(\mathbf{1}) = \mathbf{1}$, *then* $M_n(C_*([0,+\infty[)) \subset C_*([0,+\infty[)$ *for every* $n \geq 1$.

Theorem

The following statements hold true:

(a) *If* $f \in C_*([0,+\infty[)$ *(in particular if* $f \in C_0([0,+\infty[)$*), then* $\lim_{n\to\infty} M_n(f) = f$ *uniformly on* $[0,+\infty[$.

(b) *If* $f \in C_b([0,+\infty[)$, *then* $\lim_{n\to\infty} M_n(f) = f$ *uniformly on compacts subsets of* $[0,+\infty[$. back

Theorem

Then, for every $n, m \geq 1$,

(a) *M_n is a positive continuous linear operator from E_m into itself and $||M_n||_{E_m} \leq 1 + d_m/n$, where d_m is a suitable positive constant.*

(b) $M_n(E_m^0) \subset E_m^0$.

Remark

$M_n(\mathbf{1}) = \mathbf{1}$, *then* $M_n(E_m^*) \subset E_m^*$.

Theorem

The following statements hold true:

(a) *For every $m \geq 1$, if $f \in E_m^*$ (in particular, if $f \in E_m^0$), then $\lim_{n\to\infty} M_n(f) = f$ in $||\cdot||_m$.*

(b) *For every $m \geq 1$, if $f \in E_m$, then $\lim_{n\to\infty} M_n(f) = f$ uniformly on compact subsets of $[0, +\infty[$.*

back

Approximation results (on L^p-spaces ($p \geq 1$))

Theorem

Fix $1 \leq p < +\infty$. *Then,* $M_n(L^p([0,+\infty[)) \subset L^p([0,+\infty[)$ *and*

$$\|M_n\|_{L^p,L^p} \leq (b_n - a_n)^{-1/p}.$$

Moreover, if $(1/(b_n - a_n))_{n\geq 1}$ *is bounded, then for every* $f \in L^p([0,+\infty[)$

$$\lim_{n\to\infty} M_n(f) = f \quad \text{in} \quad L^p([0,+\infty[).$$

back