Nonlinear order isomorphisms on sequence spaces

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Order isomorphisms

Let E and F be paritally ordered sets. An *order isomorphism* is a bijection $T : E \to F$ so that $x \leq y \iff Tx \leq Ty$.

We are interested in the case where E and F are ordered vector spaces. (Mainly sequence spaces.)

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Some references

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F. and J. Cabello Sánchez, Some preserver problems on algebras of smooth functions, Ark. Math. 48(2010), 289-300.

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Linear stability?

$E \overset{o}{\sim} F$ means there is a (nonlinear) order isomorphism between E and F.

Say that E is *order stable* if $E \overset{o}{\sim} F$ implies that F has a subspace linearly order isomorphic to E.

Example. ℓ^1 is not order stable since $\ell^1 \overset{o}{\sim} \ell^p$, $1 < p < \infty$.

Is any space order stable?

Theorem

Let E and F be quasi-Banach lattices. If there is a closed sublattice G of E such that $c_0 \overset{o}{\sim} G$, then there is a closed sublattice H of F that is linearly order and topologically isomorphic to c_0.

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Sequence spaces

ℓ^0 = space of all real sequences with natural order.
(e_i) coordinate unit vectors in ℓ^0.
A *sequence space* is a vector subspace of ℓ^0.
A sequence space E has a *generating positive cone* if any $x \in E$ can be written as $y - z$, where $0 \leq y, z \in E$.
All sequence spaces considered below will be assumed to possess generating positive cones.

Proposition

Let E and F be sequence spaces and let $T : E \to F$ be an order isomorphism. There exist increasing bijections $\Phi_i : \mathbb{R} \to \mathbb{R}$ such that $T(a_i) = (\Phi_i(a_i))$ for all $(a_i) \in E$.

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Sequence spaces

A sequence space E is *solid* if $a \leq b \leq c$ in ℓ^0 and $a, c \in E$ imply that $b \in E$.
It is *symmetric* if $(a_i) \in E$ implies that $(a_{\sigma(i)}) \in E$ for any permutation σ on $\mathbb{N}$.

Proposition

Let E and F be order isomorphic solid symmetric sequence spaces. If $E \overset{o}{\sim} F$, then there exists an increasing bijection $\Phi : \mathbb{R} \to \mathbb{R}$ such that $T(a_i) = (\Phi(a_i))$ for all $(a_i) \in E$ is an order isomorphism from E onto F.

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Extremal cases

Proposition

Let E be a sequence space. The following are equivalent.

1. $E = \ell^0$ or c_{00},
2. For any sequence space F, $E \overset{\circ}{\sim} F$ implies $E = F$ as sets.

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If E is a solid symmetric sequence space and $E \overset{\circ}{\sim} c_0$, then $E = c_0$ as sets.

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If E is a solid symmetric sequence space and $E \overset{o}{\sim} c_0$, then $E = c_0$ as sets.

A φ function is a function $M : [0, \infty) \to \mathbb{R}$ such that $M(0) = 0$, $\lim_{t\to\infty} M(t) = \infty$, M is strictly increasing and continuous.

A φ function M satisfies the Δ_2-condition (at 0) if $\limsup_{t\to 0} M(2t)/M(t) < \infty$.
M is *dilatory* if M^{-1} satisfies the Δ_2-condition, equivalently, if there exists $K < \infty$ such that $M(Kt) \geq 2M(t)$ for all $t \geq 0$.
In particular, any convex φ function is dilatory.

The Orlicz sequence space ℓ_M consists of all $(a_n) \in \ell^0$ such that $\sum M(|a_n|/\rho) < \infty$ for some $0 < \rho \in \mathbb{R}$.

h_M is the subspace of ℓ_M consisting of all $(a_n) \in \ell^0$ such that $\sum M(|a_n|/\rho) < \infty$ for all $0 < \rho \in \mathbb{R}$.

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Theorem

Let M and N be φ functions.

1. $\ell_M \overset{o}{\sim} h_N \iff M$ and N are both Δ_2.
2. Assume that M and N are both dilatory. Then $h_M \overset{o}{\sim} h_N \iff \ell_M \overset{o}{\sim} \ell_N \iff N^{-1} \circ M$ and $M^{-1} \circ N$ are both Δ_2.

Characterizations of the conditions $h_M \overset{o}{\sim} h_N$ and $\ell_M \overset{o}{\sim} \ell_N$ for general φ functions M and N are also obtained.

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Order isomorphism between Orlicz sequence spaces

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Lorentz sequence space

Let $w = (w_n)$ be a strictly positive sequence such that $\sum w_n = \infty$.

The Lorentz sequence space $d(w,p), 1 \leq p < \infty$, is the space of all null sequences (a_n) such that $\sum (a_n^*)^p w_n < \infty$, where (a_n^*) is the decreasing rearrangement of $(|a_n|)$.

$d(w,p) \overset{\text{o}}{\sim} d(w,1) : (a_n) \mapsto (|a_n|^p \operatorname{sgn} a_n)$.

Problem: Characterize w, v such that $d(w,1) \overset{\text{o}}{\sim} d(v,1)$.

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Orlicz-Lorentz spaces

Let $w = (w_n)$ be as above and let G be a φ function.
The Orlicz-Lorentz (sequence) space $\Lambda_{w,G}$ is the space of all sequences (a_n) such that there exists $0 < \rho \in \mathbb{R}$ so that $\sum G(a_n^*/\rho)w_n < \infty$.

If $w = (1, 1, \dots)$, $\Lambda_{w,G} = \ell_G$.
If $G(x) = x^p$, then $\Lambda_{w,G} = d(w, p)$.

Problem: Characterize $\Lambda_{w,G} \overset{\circ}{\sim} \Lambda_{v,F}$.

[MS] S. Montgomery-Smith, Comparison of Orlicz-Lorentz spaces, Studia 1982: characterized equivalence (in quasinorm) if one of G, F is dilatory and one of them is Δ_2.

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(w_n) Lorentz sequence. Define
$w : [0, \infty) \to \mathbb{R}$, $w(x) = w_n$, $x \in [n-1, n)$;
$W(x) = \int_0^x W(t)\,dt$; $\widetilde{W}(x) = 1/W(1/x)$.

Proposition

Suppose that F and G are both dilatory. Then $\Lambda_{w,G} \overset{o}{\sim} \Lambda_{v,F}$ if and only if (i) $F^{-1} \circ \widetilde{V} \circ \widetilde{W}^{-1} \circ G$ is both Δ_2 and dilatory, and (ii) $\Lambda_{w,G} = \Lambda_{v,\widetilde{V}\circ\widetilde{W}^{-1}\circ G}$.

Equality of spaces $\Lambda_{w,G} = \Lambda_{v,F}$ may not yield equivalence of quasinorms in general. But this is true if G and F are dilatory.

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Suppose that G is Δ_2. Then $\Lambda_{w,G} \overset{o}{\sim} \Lambda_{w,x}$ $(= d(w,1))$.

Since x is both dilatory and Δ_2, [MS] kicks in.

Theorem

Suppose that F and G are both Δ_2. Then $\Lambda_{w,G} \overset{o}{\sim} \Lambda_{v,F}$ if and only if $\widetilde{V} \circ \widetilde{W}^{-1}$ is almost linear.

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$d(w,1) \overset{o}{\sim} d(v,1) \iff \Lambda_{w,x} \overset{o}{\sim} \Lambda_{v,x} \iff \widetilde{V} \circ \widetilde{W}^{-1}$ is almost linear.

Theorem

$d(w,1) \overset{o}{\sim} \ell_M$ for some dilatory Orlicz function $M \iff \widetilde{W}$ is almost linear.

($\implies$) Since $d(w,1)$ does not contain a closed sublattice linearly order isomorphic to c_0, neither does ℓ_M, by the Stability Theorem.
Hence M is Δ_2 and so $\ell_M \overset{o}{\sim} \ell^1 = d(v,1)$, where v is the constant 1 sequence.
Thus $\widetilde{W}$ is almost linear.

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($\implies$) Since $d(w,1)$ does not contain a closed sublattice linearly order isomorphic to c_0, neither does ℓ_M, by the Stability Theorem.
Hence M is Δ_2 and so $\ell_M \overset{o}{\sim} \ell^1 = d(v,1)$, where v is the constant 1 sequence.
Thus $\widetilde{W}$ is almost linear.

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“Duals" of Lorentz sequence spaces, Marcinkiewicz spaces

$(a_n) \in$

1. $m(w,\infty) \iff \sup \frac{a_n^*}{w_n} < \infty$,
2. $m_0(w,\infty) \iff \lim \frac{a_n^*}{w_n} = 0$,
3. $d^*(w,1) \iff$
$$\sup_n \frac{\sum_{i=1}^n a_i^*}{\sum_{i=1}^n w_i} < \infty.$$

Characterizations of order isomorphisms on these types of spaces are obtained.

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Thank you