A universal approach to topological invariants of Marcinkiewicz, Lorentz and Orlicz spaces.

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Introduction

- By *invariants of topological isomorphisms* of Banach rearrangement invariant (r.i.) spaces of measurable functions we understand the properties of such spaces preserved when instead of the original norm we consider an equivalent rearrangement invariant norm. Thus these properties are the properties not of a single norm but of the whole class of equivalent r.i. norms.

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- It is worth to note that such an approach allows us to consider also *isometrical invariants* of r.i. spaces. Namely, if $\mathbb{P}$ is an isometrical invariant of a r.i. space we introduce the corresponding topological invariant $\mathcal{P}$ saying that a Banach r.i. space X has property $\mathcal{P}$ induced by $\mathbb{P}$ if there is an equivalent r.i. norm on X such that equipped with this norm X has property $\mathbb{P}$.

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- Our approach allows to establish a natural correspondence between topological invariants of spaces in an allied triple. The properties that are in such a correspondence are closely related but superficially might look very different. For example, the Hardy - Littlewood property of Marcinkiewicz space (which is non-separable) corresponds to the Δ_2 condition (equivalent to separability) in Orlich space.

Modulars. Allied pairs of modulars. Symmetric modulars.

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  where σ_s is *compression/dilation operator*:
  $\sigma_s F(t) = F(s \cdot t), \ s, t \geq 0$. In this case we will write $F_1 \overset{m}{\sim} F_2$.

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  where σ_s is *compression/dilation operator*: $\sigma_s F(t) = F(s \cdot t), \; s, t \geq 0$. In this case we will write $F_1 \overset{m}{\sim} F_2$.
- If the relations of $\overset{m}{\sim}$ equivalence are satisfied only in a neighborhood of 0 or ∞ then we will speak about $\overset{m}{\sim}$ equivalence of norming functions in the corresponding neighborhood.

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- A continuous strictly concave norming function ψ on $[0, \infty)$ is called a *Marcinkiewicz function* or *M-function* if there is another norming function $\psi_*(t)$ such that

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- ψ_* is strictly concave.

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- Similarly, a continuous strictly convex norming function ϕ on $[0, \infty)$ is called *Orlicz function* or *N*-function; the *conjugate* function to an *N*-function ϕ is denoted ϕ^*. The modular of an *N*-function ϕ is denoted Φ and its elements are called *equiconvex* functions.

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- The fundamental functions of the Lorentz space Λ_φ and its Banach dual - the Marcinkiewicz space M_ψ are mutually dual functions: $\varphi = \psi_*$, and therefore there is a simple connection between their $\overset{m}{\sim}$invariants which we will discuss below.

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- To investigate the connection between $\overset{m}{\sim}$invariants of a Marcinkiewicz space M_ψ and an Orlicz space $\mathrm{L}^\star_\phi$ we will consider such pairs $(\mathrm{M}_\psi,\ \mathrm{L}^\star_\phi)$ of these spaces that the functions ψ and φ can be transformed into each other with the help of some natural involutions. We will call such pairs *allied*. The involutions we use are the following ones.

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- For any N-function ϕ we have the following equivalencies of M-functions: $I_3 I_2 \phi \overset{m}{\sim} I_1 I_3 \phi$; $I_4 I_3 I_2 \phi \overset{m}{\sim} I_4 I_1 I_3 \phi$.
  The described above involutions clearly generate the corresponding involutions of modulars. For modulars these equivalencies can be written as two *involution formulas*:
  1) $I_3 I_2 \Phi = I_1 I_3 \Phi$; 2) $I_4 I_3 I_2 \Phi = I_4 I_1 I_3 \Phi$.

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  1) $I_3 I_2 \Phi = I_1 I_3 \Phi$; 2) $I_4 I_3 I_2 \Phi = I_4 I_1 I_3 \Phi$.
- The involution identities allow us to call the M-modular $\Phi^{\frown} \ni \phi^{\frown} := I_4 I_1 I_3 \phi$ *allied* with the N-modular Φ. Every M-modular Ψ is allied with the N-modular $\Psi^{\smile} \ni \psi^{\smile} := (I_3)^{-1} I_1 I_4 \psi$, where ψ is an M-function and $\psi \in \Psi$.

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- The couples of functions $(\phi^{\frown}, \phi)$, and $(\psi, \psi^{\smile})$, as well as the couples of modulars $(\Phi^{\frown}, \Phi)$ and $(\Psi, \Psi^{\smile})$ are called *(mutually) allied*. The $\overset{m}{\sim}$-invariants of modulars in an allied pair are also called *mutually allied*. There is an easily established correspondence between allied $\overset{m}{\sim}$invariants.

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- It will be important for us that an $\overset{m}{\sim}$invariant of the Marcinkiewicz space $M_\xi[0, \infty)$ can be identified with a pair of $\overset{m}{\sim}$invariants of the Marcinkiewicz space $M_\xi(0, 1)$. It can be done in the following way.

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- On the class of M-functions (as well as N-functions) we define the *involution of symmetry* I_5:
  - $I_5\xi(t) := \frac{1}{\xi(\frac{1}{t})},\ t \in (0, \infty)$,

  which also defines the involution on the class of equiconcave (respectively, equiconvex) functions. An equiconcave function ξ and its modular Ξ are called *symmetric* if $\xi(t) \overset{m}{\sim} I_5\xi(t),\ t \in [0, \infty)$. It is easy to see that the Marcinkiewicz space $\mathrm{M}_\xi[0, \infty)$ generated by an equiconcave symmetric function ξ can be identified with the Marcinkiewicz space $\mathrm{M}_\xi(0, 1)$.

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- For an equiconcave function ξ on $[0, \infty)$ we define the functions ξ^0 and ξ^∞ as follows

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- $\xi^0(t) = \xi(t)$ if $t \in [0, 1]$ and $\xi^0(t) = I_5\xi(t)$ if $t \in [1, \infty)$.

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- $\xi^0(t) = \xi(t)$ if $t \in [0, 1]$ and $\xi^0(t) = I_5\xi(t)$ if $t \in [1, \infty)$.
- $\xi^\infty(t) = I_5\xi(t)$ if $t \in [0, 1]$ and $\xi^\infty(t) = \xi(t)$ if $t \in [1, \infty)$.

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- $\xi^\infty(t) = I_5\xi(t)$ if $t \in [0,1]$ and $\xi^\infty(t) = \xi(t)$ if $t \in [1,\infty)$.
- Both these functions are equiconcave and symmetric. ξ^0 is called the *left* (and ξ^∞ the *right*) symmetric parenthesis for ξ. Thus we have a natural correspondence between the Marcinkiewicz space $\mathrm{M}_\xi(0,\infty)$ and the pair of Marcinkiewicz spaces $\mathrm{M}_{\xi^0}(0,1)$ and $\mathrm{M}_{\xi^\infty}(0,1)$, generated by the left and the right symmetric parentheses of the equiconcave function ξ.

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- For a norming function ξ defined on $[0, \infty)$ may be introduced three kinds of "*supremal*" functions of compression/dilation over the parameter $s \geq 0$, as well as two kinds of lim sup-functions, and also the *lower* (γ) and the *upper* (δ) indices of compression-dilation.

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- $$\begin{cases} \mathfrak{S}_\xi(s) := \sup_{t\in[0,\infty)} \frac{\sigma_s\xi(t)}{\xi(t)},\ \gamma_\xi := \lim_{s\to 0} \frac{\log_2 \mathfrak{S}_\xi(s)}{\log_2 s}; \\ \delta_\xi := \lim_{s\to\infty} \frac{\log_2 \mathfrak{S}_\xi(s)}{\log_2 s}; \\ \mathfrak{S}^0_\xi(s) := \sup_{t\in[0,1], s\cdot t\in[0,1]} \frac{\sigma_s\xi(t)}{\xi(t)},\ \gamma^0_\xi := \lim_{s\to 0} \frac{\log_2 \mathfrak{S}^0_\xi(s)}{\log_2 s}; \\ \delta^0_\xi := \lim_{s\to\infty} \frac{\log_2 \mathfrak{S}^0_\xi(s)}{\log_2 s}; \\ \mathfrak{S}^\infty_\xi(s) := \sup_{t\geq 1,\ s\cdot t\geq 1} \frac{\sigma_s\xi(t)}{\xi(t)}\xi(t),\ \gamma^\infty_\xi := \lim_{s\to 0} \frac{\log_2 \mathfrak{S}^\infty_\xi(s)}{\log_2 s}; \\ \delta^\infty_\xi := \lim_{s\to\infty} \frac{\log_2 \mathfrak{S}^\infty_\xi(s)}{\log_2 s}; \\ \mathfrak{L}^0_\xi(s) := \limsup_{t\to 0} \frac{\sigma_s\xi(t)}{\xi(t)};\ \mathfrak{L}^\infty_\xi(s) := \limsup_{t\to\infty} \frac{\sigma_s\xi(t)}{\xi(t)}. \end{cases}$$

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- We can as well prove the equalities $\gamma_\varphi = \gamma^i_\varphi,\ \delta_\varphi = \delta^i_\varphi,\ i = 0, \infty.$

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- Because all the information about the $\overset{m}{\sim}$invariants of a triple of allied spaces can be expressed in terms of limit-supremal values of operators $\sigma_s (s \geq 0)$ on the norming function, it follows from the equivalencies above that to any $\overset{m}{\sim}$invariant of a Marcinkiewicz space $\mathrm{M}(0, \infty)$ correspond the pair of $\overset{m}{\sim}$invariants of the space $\mathrm{M}(0, 1)$ and vice versa. For a symmetric M-modular the $\overset{m}{\sim}$invariants in this pair coincide.

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- It follows from the involution formulas that the N-modular allied with a symmetric M-modular is itself symmetric. Therefore $\overset{m}{\sim}$invariants of the Orlicz space $\mathrm{L}^{\star}_{\phi}(0, 1)$ are in one-to-one correspondence with $\overset{m}{\sim}$invariants of the Marcinkiewicz space $\mathrm{M}_{\phi^{\frown}}(0, 1)$.

Examples of topological invariants of Marcinkiewicz and Orlicz spaces.

- **I.** Recall that an N-function ϕ and its *N-modular* Φ *satisfy* Δ_2-*condition* if ϕ is $\overset{m}{\sim}$equivalent at infinity to the function $\sigma_2\phi$.

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- It means that separability of the Orlicz space $L^*_\Phi(0,1)$ is equivalent to the fact that the allied Marcinkiewicz space $M_{\Phi^\frown}(0,1)$ has the Hardy - Littlewood property.

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- **II.** A norming function F is called *submultiplicative* (*supermultiplicative*) if there is a constant $c > 0$, such that the following inequalities hold

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- **Condition 4 above means that $\overset{m}{\sim}$invariant of submultiplicativity in the Marcinkiewicz space $\mathbf{M}_{\Psi}(0,1)$ is equivalent to the well known Δ'-condition in the Orlicz space $\mathsf{L}^{*}_{\Psi^{\smile}}(0,1)$.**

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1. An equiconcave function ψ is equimultiplicative if and only if we can find $\alpha, \beta :\ 0 \leq \alpha, \beta \leq 1$, such that the following equivalencies hold: at 0 $\psi(t) \overset{m}{\sim} t^{\alpha}$, at infinity $\psi(t) \overset{m}{\sim} t^{\beta}$.

2. For a symmetric equimultiplicative function $\alpha = \beta$.

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we can find α, β : $0 \le \alpha, \beta \le 1$,

such that the following equivalencies hold:

at 0 $\psi(t) \overset{m}{\sim} t^{\alpha}$, at infinity $\psi(t) \overset{m}{\sim} t^{\beta}$.

2. For a symmetric equimultiplicative function $\alpha = \beta$.

-

- Similar statements (with $\alpha, \beta \ge 1$) take place for equiconvex equimultiplicative functions.

Examples of topological invariants of Marcinkiewicz and Orlicz spaces.

- **IV.** A norming function F is called a *function of regular variation* or RV_α*-function* at 0 (respectively, at ∞) with parameter $\alpha,\ 0 \le \alpha \le \infty$ if $\lim_{t\to 0} \frac{\sigma_s F(t)}{F(t)} = s^\alpha$ (respectively, if $\lim_{t\to\infty} \frac{\sigma_s F(t)}{F(t)} = s^\alpha$).

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- If we speak only about one of singular points 0 or ∞, then we will apply the notation $F \in RV_\alpha^0$, respectively, $F \in RV_\alpha^\infty$. For an equiconcave function the parameter of regular variation can take values only in the interval $[0, 1]$, and for an equiconvex one α must be in the interval $[1, \infty)$.

Examples of topological invariants of Marcinkiewicz and Orlicz spaces.

- It is worth to note that for an M-function ψ RV-property is a property of the unit sphere of the Marcinkiewicz space $M_\psi(0,\infty)$; this property can be lost under an equivalent $\overset{m}{\sim}$ renormalization. A topological invariant of the Marcinkiewicz space $M_\Psi(0,\infty)$, induced by the RV_α-property, was first published for cases $\alpha = 0$ and $\alpha = 1$ independently by A. Mekler and E. Seneta, 1986, and then generalized for arbitrary $\alpha,\ 0 \leq \alpha \leq 1$ jointly by E. Abakumov and A. Mekler, 1994.

Examples of topological invariants of Marcinkiewicz and Orlicz spaces.

- **1. If for an equiconcave function ψ we have at 0**

$$\limsup_{t\to 0} \frac{\sigma_s\psi(t)}{\psi(t)} \overset{m}{\sim} s^{\alpha}$$

(respectively, if we have at ∞)

$$\limsup_{t\to\infty} \frac{\sigma_s\psi(t)}{\psi(t)} \overset{m}{\sim} s^{\alpha},$$

then there is a M-function $\psi_1 \overset{m}{\sim} \psi$, such that $\psi_1 \in RV_{\alpha}^{0}$ (respectively, $\psi_1 \in RV_{\alpha}^{\infty}$).

Examples of topological invariants of Marcinkiewicz and Orlicz spaces.

- If $\alpha \geq 1$ then based on involution formulas we can make a similar conclusion for $\overset{m}{\sim}$invariants induced by RV_α^∞.

Examples of topological invariants of Marcinkiewicz and Orlicz spaces.

- If $\alpha \geq 1$ then based on involution formulas we can make a similar conclusion for $\overset{m}{\sim}$invariants induced by RV_{α}^{∞}.
- For functions equiconcave on $[0, \infty)$ we will denote RV$\overset{m}{\sim}$invariants as mRV_{α}^{0} and mRV_{α}^{∞}, respectively. It was obtained in jointly paper by P.Dodds, B.De Pagter, A.Sedaev, E.Semenov, F. Sukochev, 2004, as well as by N.Kalton and F.Sukochev, 2008, that for $\alpha = 0$ and $\alpha = 1$ the corresponding $\overset{m}{\sim}$invariant provides criteria of existence or nonexistence of some types of singular functionals on Marcinkiewicz spaces that have the corresponding $\overset{m}{\sim}$invariant property.

Examples of topological invariants of Marcinkiewicz and Orlicz spaces.

2. A symmetric submultiplicative M-modular has mRV_α^0-invariant with an appropriate parameter α.

-

Examples of topological invariants of Marcinkiewicz and Orlicz spaces.

- **2. A symmetric submultiplicative M-modular has mRV_{α}^{0}-invariant with an appropriate parameter α.**

- **3. If an M-modular has mRV_{1}^{0}-invariant then it is submultiplicative on $[0,1]$.**

Examples of topological invariants of Marcinkiewicz and Orlicz spaces.

- **V.** Let us consider a topological invariant as following.

Examples of topological invariants of Marcinkiewicz and Orlicz spaces.

- **V.** Let us consider a topological invariant as following.
- Let ψ be an equiconvex symmetric function and δ_ψ be its upper index. It is known, S.Novikov, 1982, that for $1 < p \neq \frac{1}{\delta_\psi}$ the Marcinkiewicz space $M_\psi([0,1])$ is p-convex (or equivalently the Lorentz space $\Lambda_{\psi_*}([0,1])$ is q-concave, where $1/p + 1/q = 1$) if and only if when $p < \frac{1}{\delta_\psi}$. This fact is equivalent to the statement that for an equiconcave function ψ on $[0,\infty)$ and for $1 < p \neq \frac{1}{\delta_\psi}$ the power ψ^p is equiconcave if and only if when $p < \frac{1}{\delta_\psi}$.

Examples of topological invariants of Marcinkiewicz and Orlicz spaces.

- Let us now consider an $\overset{m}{\sim}$invariant property that ψ raised to the limit power $\frac{1}{\delta_\psi}$ also is equiconcave; such equiconcave functions, as well as their modulars are called *pseudopower functions*. (To avoid the clauses it follows from our definition of *M*-function that a power function cannot be pseudopower.)

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- An example of a pseudopower function on $[0,\infty)$ is provided by the equiconcave function $(\varphi^0)^{\frac{1}{2}}$, where φ^0 is defined on $[0,1]$ as $\varphi^0(0) := 0,\ \varphi^0(t) := -t \cdot \log_2 \frac{t}{2}$, and is extended *symmetrically* on $[0,\infty)$ as $\varphi^0(t) = \frac{1}{\varphi^0(\frac{1}{t})},\ t > 1$.

Examples of topological invariants of Marcinkiewicz and Orlicz spaces.

- As an example of a function that is *not pseudopower* we can consider the equiconcave function $(\varphi^{\infty})^{\frac{1}{2}}$, where the function $\varphi^{\infty}(t) := t \cdot \log_2 2t$ defined on $[1, \infty)$ is symmetrically extended on $(0, 1]$: $\varphi^{\infty}(t) = \frac{1}{\varphi^{\infty}(\frac{1}{t})}$, $t \in (0, 1]$.

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2. A pseudopower symmetric *M*-modular is submultiplicative at infinity.

-

Examples of topological invariants of Marcinkiewicz and Orlicz spaces.

- **VI.** We will say that the Marcinkiewicz space $M_\psi(0,1)$ has the property $\mathcal{P}^*$, if the nonincreasing function $\frac{\psi(t)}{t}$ belongs to $L^1[0,1]$. The $\overset{m}{\sim}$invariant $\mathcal{P}^*$ is obviously weaker than $(\mathcal{HLP})$-invariant. It is well known that for an M-function ψ **the inclusions** $M_\psi(0,1) \in \mathcal{P}^*$ **and** $\frac{d\psi(t)}{dt} \in L\log^+ L(0,1)$ are equivalent.

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- **VII.** It is known that the Orlicz space $\mathrm{L}^\star_\phi(0,1)$ and the Marcinkiewicz space $\mathrm{M}_\psi(0,1)$ can be equal as *sets*. A criterion for such an equality is that the following three conditions hold simultaneously :

Examples of topological invariants of Marcinkiewicz and Orlicz spaces.

- $$\mathbf{1}.\phi(\mathbf{t}) \stackrel{\mathbf{m}}{\sim} \mathbf{l_5 l_3 l_4}\psi;$$

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- $$\mathbf{1}.\phi(\mathbf{t}) \overset{\mathbf{m}}{\sim} \mathbf{I_5 I_3 I_4}\psi;$$
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- $$\mathbf{3}.\exists\ \varepsilon > \mathbf{0}:\ \sum_{\mathbf{n}\geq\mathbf{1}} \mathbf{2^n} \cdot \psi_*^{-\mathbf{1}}\Big(\varepsilon \cdot \psi_*(\mathbf{2^{-n}})\Big) < \infty.$$

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- This is the ultimate form of the result obtained by A. Sedaev after previous contributions by Rutickii and Lorentz.

The semigroup of M-modulars.

- We consider only M-modulars on the segment $[0, 1]$ (we can consider the case of symmetric M-modulars and reduce to it the general case of M-modular on $[0, \infty)$). On the set $\mathfrak{M}$ of all such M-modulars we can correctly define the following binary operation - composition $\Psi_1 \circ \Psi_2$, where the last expression represents the modular of the composition of M-functions

$$\psi_1 \circ \psi_2(t) := \psi_1\Big(\psi_2(t)\Big), \;\; t \in [0, 1].$$

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$$\psi_1 \circ \psi_2(t) := \psi_1\Big(\psi_2(t)\Big),\ \ t \in [0, 1].$$

- **The upper index of the composition of two functions equiconcave on $[0, 1]$ is less or equal than the product of upper indices of the composed functions and moreover, if these functions are power functions, pseudopower functions, or mRV^0-functions, the above mentioned inequality becomes equality.**

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- 4. *Ideal* if $[v_1 \in \mathcal{V} \ \mathrm{or/and} \ v_2 \in \mathcal{V}] \Rightarrow v_1 \circ v_2 \in \mathcal{V}$.

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- 5. An ideal $\mathcal{V}$ is called *closed* if $v_1 \circ v_2 \in \mathcal{V} \Leftrightarrow [v_1 \in \mathcal{V} \text{ or/and } v_2 \in \mathcal{V}]$.

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- 2°. **$\overset{m}{\sim}$invariant $(\mathcal{HLP})_\Lambda$ generates an ideal, and $\overset{m}{\sim}$invariant mRV_0^0 generates a closed ideal.**

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- **IV. The set $\bigcup_{0<\omega<1} mRV^0_\omega$ generates a subsemigroup. Moreover, $(\psi \in mRV^0_\alpha,\ \varphi \in mRV^0_\beta,\ 0 < \alpha, \beta < 1) \Rightarrow \psi \circ \varphi \in RV^0_{\alpha \cdot \beta}$.**

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- **IV. The set $\bigcup_{0<\omega<1} mRV_{\omega}^{0}$ generates a subsemigroup. Moreover,** $(\psi \in mRV_{\alpha}^{0},\ \varphi \in mRV_{\beta}^{0},\ 0 < \alpha, \beta < 1) \Rightarrow \psi \circ \varphi \in RV_{\alpha \cdot \beta}^{0}$.
- **V. The class of pseudopower M-modulars generates a commutative right closed subsemigroup.**

4. Natural interpretation. Bases. Tables.

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- A subset K of the naturals $\mathbb{N}$ is called *biinfinite* if both K and its complement $\mathbb{N} \setminus K$ are infinite subsets of $\mathbb{N}$.
- Strictly increasing sequence of integer nonnegative numbers $b = \{b_k\}_{0 \leq k < \infty}$, where $b_0 = 0$, is called a *natural base* if $\{b_k\}_{1 \leq k < \infty}$ is a biinfinite subset of $\mathbb{N}$. If $b = \{b_k\}_{k \geq 0}$ is a base then the biinfinite set $\mathbb{N} \setminus \{b_k\}_{k \geq 1} := \{b_{*i}\}_{i \geq 1}$ ordered as a strictly increasing sequence and complemented by 0 as the first element is called the *dual to b* base and is denoted b_*. The relation of duality defines an involution on the set of all bases.

Natural interpretation. Bases. Tables.

- A base b^1 is called $\overset{a}{\sim}$equivalent (= additive equivalent) to the base b^2 (we write: $b^1 \overset{a}{\sim} b^2$), if there is a natural number d, such that $b^1_k \leq b^2_{k+d} \leq b^1_{k+2d}$, $k \geq 1$.

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- The set of all bases that are $\overset{a}{\sim}$equivalent to some base b is called $\overset{a}{\sim}$modular of the base b and is denoted $\mathfrak{b}$. A property of a base valid for every base from its $\overset{a}{\sim}$modular is called $\overset{a}{\sim}$invariant. An example of an $\overset{a}{\sim}$invariant is provided by the following property

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- A base $b = \{b_n\}$ is called *condensing*, if there is a natural number d such that for any natural m, n we can find $n', \ n' = n'(m) > n$, such that

$$\sum_{i=n+1}^{n+m} \chi_b(i) \leq \sum_{i=n'+1}^{n'+m+d} \chi_b(i).$$

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- By the *lower table* of a natural base b (or *the table of base compression*) we will understand the following table of natural numbers with infinite number of rows and columns $\mathfrak{T}_b(n, m) := \sum_{i=b_n+1}^{b_{n+m}} \chi_b(i),\ n, m \geq 0.$

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- Similarly, by the *upper table* of the base b (or *the table of base dilation*) we will understand the following table $\mathfrak{T}^b(n,m) := b_{n+m} - b_n,\ n,m \geq 0.$
- Notice that the lower table characterizes the distribution of the base b as a subset of $\mathbb{N}$ while the upper table characterizes the distribution of the complement of b in $\mathbb{N}$ i.e. the distribution of the dual base.

Natural interpretation. Bases. Tables.

- In what follows we will consider a base $b = \{b_n\}$ corresponding to a symmetric equiconcave function, i.e. we assume that M-functions are defined on the unit interval. It is easy to see that to verify the relation of $\overset{m}{\sim}$equivalence at 0 for such functions it is enough to verify that they are $\overset{m}{\sim}$equivalent on the countable subset $\{2^{-n}\}_{n=0}^{\infty}$. We can use this fact to construct an interpretation of the functional model by a model of sequences of natural numbers. We will clarify the simple idea of such an interpretation without describing routine constructions.

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- We substitute a defined on $[0, 1]$ M-function ψ by its uniform piecewise linear approximation using the values of ψ at the points 2^{-n}. By taking integer parts of absolute values of logarithms of values of that piecewise linear approximation we can establish a correspondence between the function ψ and a biinfinite sequence of natural numbers (i.e. a base) b_ψ and vice versa.

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- Importantly, the relations of $\overset{m}{\sim}$ and $\overset{a}{\sim}$equivalence transform into each other and that means that the correspondence between the modulars $\Psi \leftrightarrow \mathfrak{b}_\psi$ is bijective.

Natural interpretation. Bases. Tables.

- We substitute a defined on $[0,1]$ M-function ψ by its uniform piecewise linear approximation using the values of ψ at the points 2^{-n}. By taking integer parts of absolute values of logarithms of values of that piecewise linear approximation we can establish a correspondence between the function ψ and a biinfinite sequence of natural numbers (i.e. a base) b_ψ and vice versa.
- Importantly, the relations of $\overset{m}{\sim}$ and $\overset{a}{\sim}$equivalence transform into each other and that means that the correspondence between the modulars $\Psi \leftrightarrow \mathfrak{b}_\psi$ is bijective.
- Similarly the $\overset{m}{\sim}$ and $\overset{a}{\sim}$invariants transform into each other.

Natural interpretation. Bases. Tables.

So regarding the question: *What is common between Hardy-Littlewood property of Marcinkiewicz space* $M_\psi(0,1)$ *and separability of its allied space* $L^*_\phi(0,1)$ the answer may be formulated as following: *The differences between neighboring members, - next and previous, - in each of their* $\overset{a}{\sim}$*common bases* $b_\psi \overset{a}{\sim} b_\phi$ *are bounded.*

Natural interpretation. Bases. Tables.

- Regarding the general case of a function ψ that is equiconcave on$[0, \infty)$, it is not difficult to see that it allows the interpretation by a pair of bases corresponding to the left and right symmetric parentheses of the function ψ.

Natural interpretation. Bases. Tables.

- Regarding the general case of a function ψ that is equiconcave on$[0, \infty)$, it is not difficult to see that it allows the interpretation by a pair of bases corresponding to the left and right symmetric parentheses of the function ψ.
- We can interpret all $\overset{m}{\sim}$invariants of the spaces $\mathsf{M}_\psi,\ \Lambda_\psi\ ,\text{and}\ \mathsf{L}^\star_\phi\ (\psi = \phi^\frown)$ in terms of limit and extremal relations for the upper (as well as for the lower) table of the base of an equiconcave function ψ. As examples we can consider $\overset{m}{\sim}$invariants of M-modulars on $[0, 1]$ considered in Section 2.

Natural interpretation. Bases. Tables.

- $$\textbf{I.}\ 1.\ \limsup_{n\geq 0} \mathfrak{T}^{b_\psi}(n,1) < \infty \Leftrightarrow \gamma_{b_\psi} > 0 \ \Leftrightarrow \psi^{\smile} \in (\Delta_2).$$

Natural interpretation. Bases. Tables.

- $$\textbf{I.}\ 1. \limsup_{n\geq 0} \mathfrak{T}^{b_\psi}(n,1) < \infty \Leftrightarrow \gamma_{b_\psi} > 0 \ \Leftrightarrow \psi^{\smile} \in (\Delta_2).$$
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Natural interpretation. Bases. Tables.

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- $$2. \limsup_{n\geq 0} \mathfrak{T}^{b_{\psi_*}}(n,1) < \infty \Leftrightarrow \delta_{b_\psi} < 1$$
- $$\textbf{II.}\ \ \textbf{Submultiplicativity}\ \psi : \sup_{m,n\geq 1} \Big(\mathfrak{T}^{b_\psi}(0,m) - \mathfrak{T}^{b_\psi}(n,m)\Big) < \infty.$$

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- **III. Equimultiplicativity of ψ is equivalent to any of the following conditions**

$$\sup_{n;\ k,\ m} |\mathfrak{T}^{b_\psi}(k, m) - \mathfrak{T}^{b_\psi}(n, m)| < \infty$$

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$$\psi(t) \overset{m}{\sim} t^{\gamma_{b_\psi}}.$$

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- **Repeating limit**

$$\lim_{n\to\infty} \lim_{m\to\infty} \frac{m}{\mathfrak{T}^{b_\psi}(n,m)} = \alpha \Leftrightarrow \psi \in RV^0_\alpha,\ \alpha \in [0,1].$$

Natural interpretation. Bases. Tables.

V. ψ **pseudopower** $\Leftrightarrow \sup_{n\geq 0} \mathfrak{T}^{b_{\psi_*}}(n, m) \overset{a}{\sim} \limsup_{n\to\infty} \mathfrak{T}^{b_{\psi_*}}(n, m)$.

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$$\textbf{VII. } \mathsf{M}_\psi(\mathbf{0},\mathbf{1}) = \mathsf{L}^*_\phi(\mathbf{0},\mathbf{1}) \Leftrightarrow$$

$$\exists\varphi \overset{\mathbf{m}}{\sim} \psi : \mathbf{b}_{\varphi_*} \overset{\mathbf{a}}{\sim} \mathbf{b}_{\phi^{-1}} \text{ and} \sum_{\mathbf{n}=\mathbf{0}}^{\infty} \mathbf{2}^{-\mathfrak{T}^{\mathbf{b}_{\varphi_*}}(\mathbf{n},\mathbf{1})} < \infty.$$

Natural interpretation. Bases. Tables.

- Thus, in the case of the interval $[0,1]$ all the information about topological invariants of a Marcinkiewicz space, the dual to it Lorentz space, and the allied to it Orlicz space is contained in limit and extremal values of the distribution of some natural sequence - their common base (and the dual base as well).

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- In the case of Marcinkiewicz, Lorentz, and Orlicz spaces on $[0,\infty)$, the information is contained in the corresponding values of both (left and right) bases.
- The converse is true as well. By choosing a biinfinite sequence of naturals (or a pair of such sequences) we define the corresponding allied triple of Marcinkiewicz, Lorentz, and Orlicz spaces. Moreover, by varying the distributions of this sequence in natural series we can arbitrary vary the topological invariants of the allied triple of spaces.