Modulus of non-semicompact convexity

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Contents

The talk will consist of two parts. In the first part, we develop the cornerstone theorem given in [B. de Pagter & A.R. Schep, *Measures of non-compactness of operators on Banach lattices.* J. Funct. Anal. 78:31-55, (1988), Proposition 2.1], which states that for a Banach lattice E with order continuous norm (OCN) if D is a PL-compact subset of E then $\chi(D) = \rho(D)$, by showing that if E has OCN, then $\omega(D) \leq \rho(D)$; on the other hand, if E has the Schur property, then $\rho(D) \leq \omega(D)$ for any norm bounded subset D of E. Here, χ, ρ, and ω are Hausdorff measure of non-compactness, the measure of non-semicompactness introduced in the paper above, and the measure of weak non-compactness, respectively.

Secondly, we introduce the notion of the modulus of non-semicompact convexity in Banach lattices defined with the help of the measure of non-semicompactness in Banach lattices. We extend the results obtained in [Y. Banas, *On modulus of noncompact convexity and its properties.* Canad. Math. Bull. 30:186-192, (1987)] by showing that the modulus of non-semicompact convexity is continuous and has some extra properties in reflexive Banach lattices.

Throughout the talk the letters X and Y denote infinite dimensional Banach spaces while the letters E and F are infinite dimensional Banach lattices. Let $B(x, r)$ and $S(x, r)$ denote the ball and the sphere centered at x and of radius r. For brevity we will write B_X and S_X instead of $B_X(0, 1)$ and $S_X(0, 1)$ (or B_E and S_E instead of $B_E(0, 1)$ and $S_E(0, 1)$).

If Ω is a subset of X, then $\overline{\Omega}$, $\overline{\Omega}^w$, $co(\Omega)$, $\text{dist}(x, \Omega)$ will denote the norm closure, the weak closure, the closed convex hull of Ω, and the distance from a point x to Ω, respectively. By $B(\Omega, r)$ we denote the ball centered a set Ω and with radius r, i.e., $B(\Omega, r) = \bigcup_{x \in \Omega} B(x, r)$.

We shall denote the measure of noncompactness as MNC for brevity.

Kuratowski and Hausdorff MNCs

In this section, we refer to the following two books for details and proofs.

- Akhmerov, R.R., Kamenskii, M.I., Potapov, A.S., Rodkina, A.E., Sadovskii, B.N.: Measures of Noncompactness and Condensing Operators (in Russian), Nauka, Novosibirsk, 1986; English translation: Birkhauser Verlag, 1992.
- Banas, J., Goebel, K.: Measures of Noncompactness in Banach Spaces, Marcel Dekker, 1980.

The Kuratowski MNC

If (M, d) is a metric space and Ω is a bounded subset of M, then K. Kuratowski [Kuratowski, K.: Sur les espaces complets, Fund. Math. 15, 310-309, 1930] defined $\alpha(\Omega)$, the **Kuratowski MNC** of Ω, by

$$\alpha(\Omega) = \inf\{\delta > 0 : \ \Omega = \bigcup_{i=1}^{n} \Omega_i, \exists\, \Omega_i \subset M, \ \mathrm{diam}(\Omega_i) \leq \delta, \ 1 \leq i \leq n\}$$

Also see [K. Kuratowski, Topology, Vol.1, Academic Press, 1966, p.412].
As usual, the diameter of a bounded set $S \subset M$ is defined by $\mathrm{diam}(S) = \sup\{d(x, y) : \ x, y \in S\}$.

The Hausdorff MNC

Definition

Let Ω be a norm bounded subset in X.

$$\chi(\Omega) = \inf\{\varepsilon > 0 : \Omega \textit{ has a finite } \varepsilon\textit{-net in } X\}$$

is called the Hausdorff measure of noncompactness of the set $\Omega \subset X$.

Recall that a set $S \subset X$ is called an ε-**net** of Ω if

$$\Omega \subset S + \varepsilon B_X \equiv \{s + \varepsilon b : s \in S, b \in B_X\}.$$

The Hausdorff MNC

The definition of Hausdorff MNC was given by [I.T. Gohberg, L.S. Goldenstein and A.S. Markus, *Investigation of some properties of bounded linear operators in connection with their q-norms*(Russian), Uch. Zap. Kishinevsk. Un-ta 29, 29-36, 1957].

The terminology is, indeed, motivated by the following observations. For $x \in X$, $r > 0$, define $d(S, U) = \inf\{r > 0 : S \subset U + rB_X\}$, and set $D(S, U) = \max\{d(S, U), d(U, S)\}$, the Hausdorff metric. If $\mathcal{F} = \{U \subset X : U \neq 0, \overline{U} \text{ compact}\}$ then for $S \subset X$ bounded,

$$\chi(S) = \inf_{U \in \mathcal{F}} D(S, U).$$

Equivalently, we can rewrite the Hausdorff MNC as follows.

$$\chi(\Omega) = \inf\{\delta > 0 : \Omega \subseteq \bigcup_{i=1}^{n} B(x_i, \delta), \exists\, x_1, ...x_n \in E\}.$$

Meaning of α and χ

These two measures of non-compactness are commonly used; these associate numbers to sets in such a way that compact sets all get the measure 0, and other sets get measures that are bigger according to “how far" they are removed from compactness.The underlying idea is the following: a bounded set can be covered by a single ball of some radius. Sometimes several balls of a smaller radius can also cover the set. A compact set in fact can be covered by finitely many balls of arbitrary small radius, because it is totally bounded. So one could ask: what is the smallest radius that allows to cover the set with finitely many balls?

The Hausdorff MNC in the spaces ℓ_p and c_0

Example

In the spaces ℓ_p and c_0 of sequences summable in the p-th power and respectively sequences converging to zero the MNC χ can be computed by means of the formula

$$\chi(\Omega) = \lim_{n\to\infty} \sup_{x\in\Omega} \|(I - P_n)x\|,$$

where P_n is the projection onto the linear span of the first n vectors in the standard basis.

The Hausdorff MNC in the space $C[a,b]$

Example

In the space $C[a,b]$ of continuous real valued functions on the segment $[a,b]$ the value of the set-function χ on a bounded set Ω can be computed by means of the formula

$$\chi(\Omega) = \frac{1}{2} \lim_{\delta \to 0} \sup_{x \in \Omega} \max_{0 \le \tau \le \delta} \|x - x_\tau\|,$$

where x_τ denotes the τ-translate of the function x

$$x_\tau(t) = \begin{cases} x_\tau(t+\tau) & \text{if } a \le t \le b - \tau, \\ x(b) & \text{if } b - \tau \le t \le b. \end{cases}$$

Common properties of α and χ

Let ψ be α or χ.
(a) $\psi(\Omega) = 0$ iff Ω is relatively compact. (**regularity**)
(b) $\psi(\{x\}) = 0$ (**nonsingularity**)
(c) $\Omega_1 \subseteq \Omega_2 \Rightarrow \psi(\Omega_1) \leq \psi(\Omega_2)$ (**monotonicity**)
(d) $\psi(\Omega_1 + \Omega_2) \leq \psi(\Omega_1) + \psi(\Omega_2)$ (**algebraic semi-additivity**)
(e) $|\psi(\Omega_1) - \psi(\Omega_2)| \leq L_\psi D(\Omega_1, \Omega_2)$, where $L_\chi = 1$, $L_\alpha = 2$, and D denotes the Hausdorff metric. (**Lipschitzianity**)
(f) For any $\Omega \subset E$ and any $\epsilon > 0$, $\exists\, \delta > 0$ s.t. $\forall\, \Omega_1$, $D(\Omega, \Omega_1) < \delta \Rightarrow |\psi(\Omega) - \psi(\Omega_1)| < \epsilon$. (**continuity**)
(g) $\psi(\lambda\Omega) = |\lambda|\psi(\Omega), \forall\, \lambda$. (**semi-homogeneity**)
(h) $\psi(\Omega) = \psi(\text{co}(\Omega))$. (**invariance under convex hull**)
(i) $\psi(\Omega + \{x_0\}) = \psi(\Omega)$ for any $x_0 \in E$. (**invariance under translations**)
(j) $\psi(\Omega) = \psi(\overline{\Omega})$ (**invariance under closure**)
(k) $\psi(\Omega_1 \cup \Omega_2) = \max\{\psi(\Omega_1), \psi(\Omega_2)\}$ (**semi-additivity**)

More properties

Theorem (1)

(a) $\alpha(B(x,r)) = \alpha(S(x,r)) = 2r$ *for* $r \geq 0$
(b) $\chi(B(x,r)) = \chi(S(x,r)) = r$ *for* $r \geq 0$
(c) $\chi(\Omega) \leq \alpha(\Omega) \leq 2\chi(\Omega)$
(d) $\chi(B(\Omega,r)) = \chi(\Omega) + r$ *for* $r \geq 0$.

Remark

We do not know if the equality

$$\alpha(B(\Omega,r)) = \alpha(\Omega) + 2r$$

is true for $r \geq 0$.

Homogeneous MNC

J. Mallet-Paret and R.D. Nussbaum developed in [J. Mallet-Paret, R.D. Nussbaum, *Inequivalent measures of noncompactness and the radius of the essential spectrum* PAMS, 2011] the concepts of Hausdorff and Kuratowski MNCs which share most of the properties of them.

Definition

Let $(X, \|.\|)$ be a real or complex Banach space and let $\mathcal{B}(X)$ be the collection of all bounded subsets of X. A map $\beta : \mathcal{B}(X) \to [0, \infty)$ is a ***homogeneous MNC on*** X *if β satisfies properties (a)-(j), with β replacing ψ in these conditions and β is a* ***homogeneous, set-additive MNC*** *if β satisfies properties (a)-(k), with β replacing ψ in these conditions.*

For example, χ, Hausdorff measure of non-compactness on X, is a homogeneous, set-additive MNC.

Homogeneous MNC

Two homogeneous MNC β and γ on X are "equivalent" if there exist positive constants b and c scuh that $b\beta(S) \leq \gamma(S) \leq c\beta(S)$ for all bounded sets $S \subset X$. If such constants do not exist, the measures of noncompactness are "inequivalent". They ask in the later paper [J. Mallet-Paret, R.D. Nussbaum, *Inequivalent measures of noncompactness* Annali di Mathematica, 2011] a foundational question: For what infinite dimensional Banach spaces do there exist inequivalent measures of noncompactness on X? They give some examples of inequivalent measures of noncompactness. For instance, they prove that such inequivalent measures of noncompactness exist if X is a Hilbert space.

[F.S. De Blasi, *On a property of the unit sphere in a Banach space*, Bull. Math. Soc. Sci. Math., 259-262, 1977]

Definition

Let X be a Banach space. The function $\omega : 2^X \to [0, \infty)$, defined as

$$\omega(\Omega) = \inf\{\varepsilon > 0 : \Omega \text{ has a weakly compact } \varepsilon\text{-net in } X\},$$

is called ***the measure of weak non-compactness.***

The measure of weak non-compactness is an MNC in the sense of the general definition provided X is endowed with the weak topology. Namely, it is defined by

$$\omega(\Omega) = \inf\{\varepsilon > 0 : \exists\, W \text{ (weakly compact)} \subset X \text{ s.t. } \Omega \subseteq W + \delta B_X\}$$

Properties of ω

(a) $\omega(\Omega) = 0$ iff Ω is relatively weakly compact. (**regularity**)
(b) $\Omega_1 \subseteq \Omega_2 \Rightarrow \omega(\Omega_1) \leq \omega(\Omega_2)$ (**monotonicity**)
(c) $\omega(\Omega_1 + \Omega_2) \leq \omega(\Omega_1) + \omega(\Omega_2)$ (**algebraic semi-additivity**)
(d) $\omega(\lambda\Omega) = |\lambda|\omega(\Omega), \forall\, \lambda$. (**semi-homogeneity**)
(e) $\omega(\Omega) = \omega(\text{co}(\Omega))$. (**invariance under convex hull**)
(f) $\omega(\Omega + \{x_0\}) = \omega(\Omega)$ for any $x_0 \in E$. (**invariance under translations**)
(g) $\omega(\Omega) = \omega(\overline{\Omega}^w)$ (**invariance under weak closure**)
(h) $\omega(\Omega_1 \cup \Omega_2) = \max\{\omega(\Omega_1), \omega(\Omega_2)\}$ (**semi-additivity**)

More properties

Theorem

(a) *If X is reflexive, then* $\omega(B_X) = 0$*.*
(b) *If X is non-reflexive, then* $\omega(B_X) = 1$*.*
(c) $\omega(\Omega) \leq \chi(\Omega) \leq \alpha(\Omega)$*.*
(d) $\omega(\Omega + rB_X) = \omega(\Omega) + r\,\omega(B_X), r \geq 0$*.*

[B. de Pagter, A.R. Schep, *Measures of non-compactness of operators in Banach lattices*, J. Funct. Anal. 78, 1988]

Given $x \in E$, the principal ideal $E_x = \{z \in E : |z| \leq n|x| \text{ for some } n \in N\}$ is, by the Yosida representation theorem [LZ, Theorem 45.4], isomorphic to a space $C(K)$ of all complex valued continuous functions on a compact Hausdorff space K, such that the element $|x|$ corresponds to the function 1.

The multiplication by x in $C(K)$ now induces a corresponding operator in E_x, which will be denoted by M_x. Then $M_x(|x|) = x$ and $|M_x(z)| = |z|$ for all $z \in E_x$. Note that $M_x \in Z(E_x)$, the center of E_x, and $|M_x| = I$. In general, M_x cannot be extended to the whole space E. If E is Dedekind complete, however, then M_x, can be extended to an element of the center $Z(E)$ of E (with absolute value equal to I).

Almost Order Bounded Set

Definition

*A subset D of E is called **almost order bounded** if for every $\varepsilon > 0$ there exists $0 \le u \in E$ such that $\|(|x| - u)^+\| \le \varepsilon$ for all $x \in D$ (see [Z, Sect. 122]).*

Proposition

$\|(|x| - u)^+\| \le \varepsilon$ *for all* $x \in D$ *if and only if* $D \subseteq [-u, u] + \varepsilon B_E$.

The Measure of Non-semicompactness

Definition

Let $D \subseteq E$ be a norm bounded set.

$$\rho(D) = \inf\{\delta > 0 : \exists 0 \leq u \in E \text{ s.t. } \|(|x| - u)^+\| \leq \delta, \forall x \in D\}$$

is called the measure of non-semicompactness of the set D.

From the preceding Proposition it is clear that

$$\rho(D) = \inf\{\delta > 0 : \exists 0 \leq u \in E \text{ s.t. } D \subseteq [-u, u] + \delta B_E\}.$$

Properties of ρ

(a) $\rho(D) = 0$ iff Ω is almost order bounded. (**regularity**)
(b) $\rho(D_1 + D_2) \leq \rho(D_1) + \rho(D_2)$ (**algebraic semi-additivity**)
(c) $\rho(\lambda D) = |\lambda|\rho(D), \forall\, \lambda$. (**semi-homogeneity**)
(d) $D_1 \subseteq D_2 \Rightarrow \rho(D_1) \leq \rho(D_2)$ (**monotonicity**)
(e) $\rho(D) = \rho(\overline{D})$ (**invariance under closure**)
(f) $\rho(D) \leq \chi(D)$ (Indeed, $D \subseteq \bigcup_{j=1}^{n} B(f_j, \delta)$ implies that $\|(|f| - u)^+\| \leq \delta$ for all $f \in D$ with $u = \sup(|f_1|, ..., |f_n|)$.)

Remark

We assert that $\rho(B_E) < 1$ *if and only if there exists an order unit* $0 \leq u \in E$ *and the norm in* E *is equivalent to the order unit norm* $\|.\|_u$. *Indeed, suppose* $\rho(B_E) < \delta < 1$, *then there exists* $0 \leq u \in E$ *such that* $B_E \subseteq [-u, u] + \delta B_E$. *Now it follows that* $B_E \subseteq [-u, u] + \delta[-u, u] + ... + \delta^n[-u, u] + \delta^{n+1} B_E$, *and hence*

$$B_E \subseteq (1-\delta)^{-1}[-u, u] + \delta^{n+1} B_E.$$

Letting $n \to \infty$ *we get* $B_E \subseteq (1-\delta)^{-1}[-u, u]$, *and the result follows. We note that thus*

$$\rho(B_E) = 0 \text{ or } 1.$$

For any $0 \leq \phi \in E*$ we define the Riesz seminorm p_ϕ on E by $p_\phi(f) = \langle |f|, \phi \rangle$. Furthermore, for $f \in E$ and $\varepsilon > 0$ we denote $B_\phi(f, \varepsilon) = \{g \in E : p_\phi(f - g) \leq \varepsilon\}$. The set $D \subseteq E$ is called **PL-compact** if for every $0 \leq \phi \in E*$ and every $\varepsilon > 0$ there exist $f_1, ..., f_n \in E$ such that

$$D \subseteq \bigcup_{j=1}^{n} B_\phi(f_j, \varepsilon)$$

[5, Definition 4.11]. Observe that D is PL-compact if and only if D is p_ϕ-precompact for every $0 \leq \phi \in E^*$ [Z, Definition 124.7].

Theorem

Let E be a Banach lattice with order continuous norm. If $D \subseteq E$ is PL-compact, then $\chi(D) = \rho(D)$.

A property for ρ

Proposition

$\rho(D + rB_E) = \rho(D) + r\rho(B_E)$ *for* $r \geq 0$.

Proof. Clearly, $\rho(D + rB_E) \leq \rho(D) + r\rho(B_E)$. There are only two possibilities: $\rho(B_E)$ is 0 or 1.
If $\rho(B_E) = 0$, then $\rho(D + rB_E) \leq \rho(D)$. Take $\delta > \rho(D + rB_E)$. Then $\exists\, 0 \leq u \in E$ s.t. $D + rB_E \subset [-u, u] + \delta B_E$. In this case, $D \subset [-u, u] + (\delta - r)B_E \subset [-u, u] + \delta B_E$, and hence, $\rho(D) \leq \delta$. So, $\rho(D) \leq \rho(D + rB_E)$.
If $\rho(B_E) = 1$, then $\rho(D + rB_E) \leq \rho(D) + r$. We first observe that $r \leq \rho(D + rB_E)$. Otherwise, if $x \in D$, then $rB_E \subset D \setminus \{x\} + rB_E$, and then
$r = r\rho(B_E) = \rho(rB_E) \leq \rho(D \setminus \{x\} + rB_E) = \rho(D + rB_E) < r$, which is a contradiction.

A property for ρ

Now by definition of $\rho(D + rB_E)$, we have $0 \leq u \in E$ and $\exists\, \delta > \rho(D + rB_E)$ s.t. $D + rB_E \subset [-u, u] + \delta B_E$. Then

$$D + rB_E \subset \overline{\text{co}}([-u, u]) + \delta B_E = \overline{\text{co}}([-u, u]) + (\delta - r)B_E + rB_E$$

$$= [-u, u] + (\delta - r)B_E + rB_E.$$

Theorem (H. Radstrom, An embedding theorem for spaces of convex sets, PAMS 3, 165-169, 1952)

Let X, Y, Z be non-empty subsets of a Banach space. If Y is closed and convex, Z is bounded, and $X + Z \subset Y + Z$, then $X \subset Y$.

By using the above theorem, since $[-u, u] + (\delta - r)B_E$ is closed and convex, it follows that $D \subset [-u, u] + (\delta - r)B_E$. Then $\rho(D) \leq \delta - r$. This implies that $\rho(D) + r \leq \delta$, and hence, $\rho(D) + r \leq \rho(D + rB_E)$. $\square$

Comparison of ρ and ω

Definition

A Banach space has the ***Schur property*** *if every weak convergent sequence is norm convergent.*

Lemma (1)

If E is a Banach lattice with Schur property, then $\rho(D) \leq \omega(D)$.

Proof. Take $\delta > \omega(D)$. $\exists\, W$ (weakly compact) s.t. $D \subset W + \delta B_E \subset \text{sol}(W) + \delta B_E$.

Let $\varepsilon > 0$. Since W is weakly compact, it follows from [AB, Theorem 13.3] that every disjoint sequence in sol(W) converges weakly to zero. But E has the Schur property; so the convergence is in norm, that is, every disjoint sequence in sol(W) is norm convergent to zero.

Comparison of ρ and ω

Applying [AB, Theorem 13.5] to the identity operator $I : E \to E$, $\rho(x) = \|x\|$, and the solid hull of W, we see that $\exists 0 \leq u \in E$ (lying in the ideal generated by W) s.t. $\|(|x| - u)^+\| < \varepsilon$ holds for all $x \in W$.

From $|x| = |x| \wedge u + (|x| - u)^+$ and [AB, Theorem 1.9], we see that $\mathrm{sol}(W) \subset [-u, u] + \varepsilon B_E$. Thus, $D \subset \mathrm{sol}(W) + \delta B_E \subset [-u, u]$, and so $\rho(D) \leq \delta + \varepsilon$. Since ε was arbitrary, it follows that $\rho(D) \leq \delta$. Hence, $\rho(D) \leq \omega(D)$. $\square$

Comparison of ρ and ω

Lemma (2)

If E is a Banach lattice with order continuous norm, then $\omega(D) \leq \rho(D)$.

Proof. Take $\delta > \rho(D)$. Then $\exists\, 0 \leq u \in E$ s.t. $D \subset [-u, u] + \delta B_E$. Since E has order continuous norm, it follows from [AB, Theorem 12.9] that each order interval of E is weakly compact. So, $[-u, u]$ is weakly compact, and hence, $\omega(D) \leq \delta$ Thus, $\omega(D) \leq \rho(D)$. □

From Lemma 1 and Lemma 2, we obtain the following result.

Theorem

If E is a Banach lattice with Schur property and order continuous norm, then $\rho(D) = \omega(D)$.

Goal

The classical modulus of convexity introduced by J.A. Clarkson in 1936 to define uniformly convex spaces is at the origin of a great number of moduli defined since then [Alanso, J., Ullan, A.: Moduli of convexity, Functional Analysis and approximation, Ed. by P.L. Papini, Bagni di Lucca Italy, 16-20 May, 1988]. Indeed, there are a lot of quantitative descriptions of geometrical properties of Banach spaces. The most common way for creating these descriptions is to define a real function (a "modulus") depending on the Banach space under consideration, and from this a suitable constant or coefficient closely related with this function. The moduli and/or the constants are attempts in order to get a better understanding about the fact: the shape of the unit ball of a space.

Goal

One might well ask: Are there too many moduli for these purposes? Maybe! In part this is because many of these moduli involve very difficult computations, and, often there intricate links between them. Moreover, it is not unusual to find some moduli defined in (seemingly) different way, depending on the preferences.

The aim of this part of our study is to give the modulus of non-semicompact convexity for Banach lattices with some properties.

Clarkson modulus of convexity [Clarkson 36]

It is the function $\delta_X : [0,2] \to [0,1]$ given by

$$\delta_X(\varepsilon) := \inf\{1 - \|\frac{x+y}{2}\| : \ x, y \in B_X, \ \|x-y\| \geq \varepsilon\}.$$

The *characteristic (or coefficient) of convexity of* X,

$$\varepsilon_0(X) = \sup\{\varepsilon \in [0,2] : \ \delta_X(\varepsilon) = 0\}$$

was defined in [Goebel 70].
X is called *uniformly convex* if $\varepsilon_0(X) = 0$ (i.e.,

$$\forall\, 0 < \varepsilon \leq 2, \exists\, \delta > 0 \text{ s.t. } \forall x, y \in B_X, \|\frac{x+y}{2}\| \leq 1 - \delta \Rightarrow \|x-y\| \geq \varepsilon.)$$

Goebel-Sekowski modulus of noncompact convexity [Goebel-Sekowski 84]

The modulus of noncompact convexity of X (w.r.t. α) is the function $\Delta_\alpha : [0,2] \to [0,1]$ defined by

$$\Delta_\alpha(\varepsilon) := \inf\{1 - \mathrm{dist}(0, A) : A \subset B_X, A = \overline{co}(A), \alpha(A) \geq \varepsilon\}$$

$$= \inf\{1 - \inf\{\|x\| : x \in A\} : A(\text{convex}) \subset B_X, \alpha(A) \geq \varepsilon\}.$$

- Δ_α is non-decreasing.
- For any X, $\delta_X(\varepsilon) \leq \Delta_\alpha(\varepsilon)$.

The coefficient of noncompact convexity is

$$\varepsilon_\alpha(X) = \sup\{\varepsilon : \Delta_\alpha(\varepsilon) = 0\}.$$

X is called Δ_α-uniformly convex if $\varepsilon_\alpha(X) = 0$.

- $\varepsilon_\alpha(X) \leq \varepsilon_0(X)$.

Banas modulus of noncompact convexity [Banas 87]

The modulus of noncompact convexity of X (w.r.t. χ) is the function $\Delta_\chi : [0,1] \to [0,1]$ defined by

$$\Delta_\chi(\varepsilon) := \inf\{1 - \text{dist}(0, A) : A \subset B_X, A = \overline{co}(A), \chi(A) \geq \varepsilon\}$$

$$= \inf\{1 - \inf\{\|x\| : x \in A\} : A(\text{convex}) \subset B_X, \chi(A) \geq \varepsilon\}.$$

- Δ_χ is non-decreasing.
- For any X, $\delta_X(\varepsilon) \leq \Delta_\alpha(\varepsilon) \leq \Delta_\chi(\varepsilon) \leq \Delta_\alpha(2\varepsilon)$, $0 \leq \varepsilon \leq 1$, since $\chi(A) \leq \alpha(A) \leq 2\chi(A)$.

The coefficient of noncompact convexity is

$$\varepsilon_\chi(X) = \sup\{\varepsilon : \Delta_\chi(\varepsilon) = 0\}.$$

X is called Δ_χ-uniformly convex if $\varepsilon_\chi(X) = 0$.

- $\varepsilon_\chi(X) \leq \varepsilon_\alpha(X) \leq 2\varepsilon_\chi(X)$.

Modulus of non-semicompact convexity

The modulus of non-semicompact convexity of E (w.r.t. ρ) is the function $\Delta_\rho : [0,1] \to [0,1]$ defined by

$$\Delta_\rho(\varepsilon) := \inf\{1 - \operatorname{dist}(0, A) : A \subset U_E, A = \overline{co}(A), \rho(A) \geq \varepsilon\}$$

$$= \inf\{1 - \inf\{\|x\| : x \in A\} : A(\text{convex}) \subset U_E, \rho(A) \geq \varepsilon\}.$$

- Δ_ρ is non-decreasing.
- For any X, $\delta_X(\varepsilon) \leq \Delta_\alpha(\varepsilon) \leq \Delta_\chi(\varepsilon) \leq \Delta_\rho(\varepsilon)$, $0 \leq \varepsilon \leq 1$, since $\rho(A) \leq \chi(A) \leq \alpha(A)$.

The coefficient of non-semicompact convexity is

$$\varepsilon_\rho(E) = \sup\{\varepsilon : \Delta_\rho(\varepsilon) = 0\}.$$

E is called Δ_ρ-uniformly convex if $\varepsilon_\rho(E) = 0$.

- $\varepsilon_\rho(X) \leq \varepsilon_\chi(X) \leq \varepsilon_\alpha(X) \leq \varepsilon_0(X)$.

Continuity of Modulus of non-semicompact convexity

Theorem

Δ_ρ is continuous on $[0,1]$.

Proof. We know that Δ_ρ is decreasing. Fix $0 \le \varepsilon < 1$ and take arbitrary $\varepsilon_1 < \varepsilon_2 < 1$. For sufficiently small $\eta > 0$ we may choose an A_1 (convex) $\subset B_E$, $\rho(A_1) \ge \varepsilon_1$ and

$$1 - \mathrm{dist}(0, A_1) \le \Delta_\rho(\varepsilon_1) + \eta \qquad (*)$$

Say $\frac{1-\varepsilon_2}{1-\varepsilon_1} =: k$. Then $0 < k < 1$.
If $kA_1 =: Y$, then $\rho(Y) = k\rho(A_1)$, $\mathrm{dist}(0, Y) = k\mathrm{dist}(0, A_1)$, $\mathrm{dist}(Y, S) \ge 1 - k$.
If $B(A_1, 1-k) =: A_2$, then A_2 (convex) $\subset B_E$,

$$\mathrm{dist}(0, A_2) = k\mathrm{dist}(0, A_1) - 1 + k \qquad (**)$$

Continuity of Modulus of non-semicompact convexity

It follows from the following Proposition

Proposition

$\rho(D + rB_E) = \rho(D) + r\rho(B_E)$ for $r \geq 0$.

we have $\rho(A_2) = \rho(B(A_1, 1-k)) = \rho(A_1) + (1-k)\rho(B_E) \geq k\rho(A_1) + (1-k)\rho(B_E)$.
If $\rho(B_E) = 0$, then $\rho(A_2) = \rho(A_1) \geq \varepsilon_1$.
If $\rho(B_E) = 1$, then $\rho(A_2) \geq k\varepsilon_1 + (1-k) = \varepsilon_2$. From $*$ and $**$, we get $1 - \text{dist}(0, A_2) = 1 - k\text{dist}(0, A_1) + 1 - k = k(1 - \text{dist}(0, A_1)) + 2(1-k) \leq k(\Delta_\rho(\varepsilon_1) + \eta) + 2(1-k)$.
Hence, $\Delta_\rho(\varepsilon_2) \leq k(\Delta_\rho(\varepsilon_1) + \eta) + 2(1-k)$. Since η was arbitrary, we have $\Delta_\rho(\varepsilon_2) \leq k\Delta_\rho(\varepsilon_1) + 2(1-k)$. Therefore, $\Delta_\rho(\varepsilon_2) - \Delta_\rho(\varepsilon_1) \leq k\Delta_\rho(\varepsilon_1) - \Delta_\rho(\varepsilon_1) + 2(1-k) = (1-k)(2 - \Delta_\rho(\varepsilon_1)) \leq 2(1-k) = 2\frac{\varepsilon_2 - \varepsilon_1}{1 - \varepsilon_1}$. $\square$

The case of reflexive Banach lattice

Let E be a reflexive Banach lattice. In this case, from [Köthe, TVS I], for $\emptyset \neq A$ (closed+convex) $\subset E$ and for any $x \in E$, $\exists\, a \in A$ s.t. $\text{dist}(x, A) = \|x - a\|$.

Theorem

$\Delta_\rho(k\varepsilon) \leq k\Delta_\rho(\varepsilon)$ *for any* $0 \leq k, \varepsilon \leq 1$.

Proof. Fix $0 < \varepsilon < 1$ and take $\eta > 0$, A (convex) $\subset B_E$, $\rho(A) \geq \varepsilon$,

$$1 - \text{dist}(0, A) \leq \Delta_\rho(\varepsilon) + \eta \qquad (*).$$

Let $0 < k < 1$ and choose $a \in A$ with $\text{dist}(0, A) = \|a\|$.

The case of reflexive Banach lattice

If $kA + \frac{1-k}{\|a\|}a =: A_1$, then $\rho(A_1) \geq k\varepsilon$ and $\mathrm{dist}(0, A_1) = k\mathrm{dist}(0, A) + 1 - k$. Moreover, $A_1 \subset B_E$ and

$$\mathrm{dist}(0, A) = \frac{1}{k}[\mathrm{dist}(0, A_1) + k - 1].$$

From $(*)$, $1 - \Delta_\rho(\varepsilon) \leq \mathrm{dist}(0, A) + \eta = \frac{1}{k}[\mathrm{dist}(0, A_1) + k - 1] + \eta$.

Since $\Delta_\rho(k\varepsilon) = \inf\{1 - \mathrm{dist}(0, A_1) : A_1 \text{ (convex)} \subset B_E, \rho(A_1) \geq k\varepsilon\}$,

it follows that

$$1 - \Delta_\rho(\varepsilon) \leq \frac{1}{k}[1 - \Delta_\rho(k\varepsilon) + k - 1] + \eta = 1 - \frac{1}{k}\Delta_\rho(k\varepsilon).$$

Hence, $\Delta_\rho(k\varepsilon) \leq k\Delta_\rho(\varepsilon)$. $\square$

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