Domination by ergodic elements in ordered Banach algebras

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Outline of talk

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- Overview of problems we investigated

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- Provide conditions under which a positive element in an ordered Banach algebra dominated by a positive ergodic element will be ergodic.

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- Provide conditions under which a positive element in an ordered Banach algebra dominated by a positive ergodic element will be ergodic.
- Establish an ergodic theorem, providing necessary and sufficient conditions for an element to be ergodic.

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First literature on ordered Banach algebras:

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- H. Raubenheimer and S. Rode: Cones in Banach algebras, *Indag. Math.* **7** (1996), 489 – 502.
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Some of the topics investigated in ordered Banach algebras:

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In a semisimple Banach algebra, $\alpha \in \sigma(a)$ is a Riesz point relative to an inessential ideal I if and only if α is a pole of the resolvent of a and $p(a, \alpha) \in I$.

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$\mathcal{K}^r(E)$: the closure in $\mathcal{L}^r(E)$ of the ideal of finite rank operators on E

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An algebra cone C of A is *normal* if there exists a constant $\alpha > 0$ with the following property:
if $0 \leq x \leq y$ relative to C, then $||x|| \leq \alpha ||y||$.

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Last fact proved by J. Martinez and J. M. Mazón in 1991.

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Theorem (S. Mouton, H. Raubenheimer, 1997)

Let (A, C) be an OBA with C closed and the spectral radius in (A, C) monotone, and let I be a closed inessential ideal in A such that the spectral radius in $(A/I, \pi C)$ is monotone.

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Corollary (2)

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The conditions force α to be a simple pole:

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Corollary (3)

Let A be a Banach algebra and $a \in A$. Let (f_n) be a sequence of complex valued functions analytic on a neighbourhood of $\sigma(a)$. Suppose that $\alpha \neq 0$ is a pole of order at most $k \geq 1$ of the resolvent of a. If $f_n(\alpha) \to 1$ and $(\alpha 1 - a)f_n(a) \to 0$ as $n \to \infty$, then α is a simple pole of the resolvent of a.

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Corollary (2)

Let A be a Banach algebra and $a \in A$. Let (f_n) be a sequence of complex valued functions analytic on a neighbourhood of $\sigma(a)$. Suppose that $\alpha \neq 0$ is a simple pole of the resolvent of a such that $f_n(\alpha) \to 1$ as $n \to \infty$. If $(\alpha 1 - a)f_n(a) \to 0$ as $n \to \infty$, then $f_n(a) \to p(a, \alpha)$ as $n \to \infty$.

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$$\alpha = 1; \qquad f_n(\lambda) = \sum_{k=0}^{n-1} \frac{\lambda^n}{n}$$

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Theorem (The ergodic theorem)

Let A be a Banach algebra, $a \in A$, $f_n(\lambda) = \sum_{k=0}^{n-1} \frac{\lambda^n}{n}$ $(\lambda \in \mathbb{C}$, $n \in \mathbb{N})$ and $1 \in \text{iso}\, \sigma(a)$. Then a is ergodic and $f_n(a) \to p(a, 1)$ if and only if 1 is a (simple) pole of the resolvent of a and $(1 - a) f_n(a) \to 0$ as $n \to \infty$.

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Proved for the bounded linear operators on a complex Banach space by N. Dunford in 1943, using partly operator theoretic methods.

Domination by ergodic elements

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Let (A, C) be a semisimple OBA with C closed and normal and let I be a closed inessential ideal of A such that the spectral radius in $(A/I, \pi C)$ is monotone.

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- By the ergodic theorem a is ergodic.

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Corollary

Let E be a Dedekind complete Banach lattice and let $S, T \in \mathcal{L}^r(E)$ such that $0 \leq S \leq T$. If T is uniformly ergodic with ergodic projection of finite rank, then S is uniformly ergodic.

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Proved for the bounded linear operators on a Banach lattice (under a weaker form of domination) by F. Räbiger and M. P. H. Wolff in 1997, using operator theoretic methods.

THANK YOU