COMMUTATIVELY ORDERED BANACH ALGEBRAS

K. Muzundu

University of Zambia

kmzundu@gmail.com

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Overview

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- All Banach algebras will be assumed to be complex and unital.
- All ideals will be assumed to be two sided.
- The canonical map for a quotient algebra will be denoted by π.
- Spectrum: $\sigma(x) = \{\lambda \in \mathbb{C} : x - \lambda 1 \notin A^{-1}\}$.
- Spectral radius: $r(x) = \sup\{|\lambda| : \lambda \in \sigma(x)\}$.

Definition (S. Rhode, H. Raubenheimer, 1996)

A subset C of a Banach algebra A is called an algebra cone if C satisfies the following for all $x, y \in C$ and positive scalars λ:
(i) $x + y \in C$,
(ii) $\lambda x \in C$,
(iii) $xy \in C$,
(iv) $1 \in C$.

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- A Banach algebra ordered by an algebra cone is called an ordered Banach algebra (OBA).

Example

1. $(M_2(\mathbb{C}), C)$, with $C = \{(\alpha_{ij}) \in M_2(\mathbb{C}) : \alpha_{ij} \geq 0\}$.

2. $(\mathcal{L}^r(E), K)$, where E is a Dedekind complete Banach lattice, $C = \{x \in E : x \geq 0\}$ and $K = \{T \in \mathcal{L}(E) : TC \subset C\}$.

3. (A, C), where A is a commutative C^*-algebra and $C = \{x \in A : x = x^*$ and $\sigma(x) \subseteq [0, \infty)\}$

Ordered Banach algebras

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- R. Harte: Tim Starr, Trevor West and a positive contribution to operator theory, *Filomat* **21(1)** (2007), 129–135.
- Some results proved in partially ordered Banach algebras.

Ordered Banach algebras

- **QUESTION:** Can the entire known spectral theory in OBAs be extended in this way?

Ordered Banach algebras

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- **ANSWER:** Yes - with minor adjustments for some results and major adjustments for others.

Commutatively ordered Banach algebras

Definition (S. Mouton, K. Muzundu, 2012)

A subset C of a Banach algebra A is called an algebra c-cone if C has the following properties for all $x, y \in C$ and all positive scalars λ:
(i) $x + y \in C$,
(ii) $\lambda x \in C$,
(iii) $xy \in C$ if $xy = yx$,
(iv) $1 \in C$.

- A Banach algebra ordered by an algebra c-cone is called a commutatively ordered Banach algebra (COBA).
- Every OBA is a COBA.

Commutatively ordered Banach algebras

Example (S. Mouton, K. Muzundu, 2012)

1. $(\mathcal{L}(H), C)$, where H is a Hilbert space and $C = \{T \in \mathcal{L}(H) : T \geq 0\}$.

2. (A, C), where A is a C^*-algebra and $c = \{x \in A : x^* = x$ and $\sigma(x) \subseteq [0, \infty)\}$.

3. $(A_1 \oplus A_2, C_1 \oplus C_2)$, where A_1 is an OBA and A_2 is a COBA.

Commutatively ordered Banach algebras

Definition (S. Rhode, H. Raubenheimer, 1996)

An algebra cone C in a Banach algebra A is said to be:
(i) proper if $C \cap -C = \{0\}$,
(ii) closed if C is a closed subset of A,
(iii) normal if there is a scalar $\alpha > 0$ such that $||x|| \leq \alpha ||y||$ whenever $0 \leq x \leq y$.

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Definition (S. Rhode, H. Raubenheimer, 1996)

We say that the spectral radius in an OBA (A, C) is monotone if $r(x) \leq r(y)$ whenever $0 \leq x \leq y$.

Theorem (S. Rhode, H. Raubenheimer, 1996)

Let A be a Banach algebra with an algebra cone C. The we have the following:
(a) If C is normal, then C is proper.
(b) If C is normal, then the spectral radius in (A, C) is monotone.
(c) If C is closed and the spectral radius in (A, C) is monotone, then $r(x) \in \sigma(x)$ for every $x \in C$.

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Definition (S. Mouton, K. Muzundu, 2012)

If C is an algebra c-cone in a Banach algebra A, then we say that the spectral radius is c-monotone in (A, C) if $r(x) \leq r(y)$ whenever $0 \leq x \leq y$ and $xy = yx$.

Theorem (S. Mouton, K. Muzundu, 2012)

Let A be a Banach algebra with an algebra c-cone C. Then we have the following:
(i) If C is c-normal, then C is proper
(ii) If C is c-normal, then the spectral radius in (A, C) is c-monotone
(iii) If C is closed and the spectral radius in (A, C) is c-monotone, then $r(x) \in \sigma(x)$ for all $x \in C$.

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- However, πC does have the properties:

  (i) $\pi C + \pi C \subset \pi C$,
  (ii) $\lambda(\pi C) \subset \pi C$,
  (iii) $x^n + F \in \pi C$ for all $x \in C$,
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- A Banach algebra ordered by an algebra C'-cone is called a C'OBA.

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- A Banach algebra ordered by an algebra C'-cone is called a C'OBA.
- Some of the results can be proved in the C'OBA structure.

Quotient algebras

Definition (S. Mouton, K. Muzundu, 2012)

Let (A, C) be a COBA. A subset $M \subseteq C$ is called a maximal positive commutative subset of C if M is commutative and it is not properly contained in another commutative subset of C.

Quotient algebras

Theorem (S. Mouton, K. Muzundu, 2012)

Every maximal positive commutative set M in a COBA (A, C) is an algebra cone of A.

END