Operators on ordered spaces belonging to certain ideals
(TO, E. Spinu)

Positivity VII, Leiden, July 2013

Operator ideals

$\mathcal{I}$ is an operator ideal if, for any Banach spaces X, Y,

- $\mathcal{I}(X, Y)$ is a linear subspace of $B(X, Y)$, containing all finite rank operators.
- If $T \in \mathcal{I}(X, Y)$, $U \in B(X', X)$, and $V \in B(Y, Y')$, then $VTU \in \mathcal{I}(X', Y')$.

We look for criteria for membership in $\mathcal{I}$.

Operator ideals

$\mathcal{I}$ is an operator ideal if, for any Banach spaces X, Y,

- $\mathcal{I}(X, Y)$ is a linear subspace of $B(X, Y)$, containing all finite rank operators.
- If $T \in \mathcal{I}(X, Y)$, $U \in B(X', X)$, and $V \in B(Y, Y')$, then $VTU \in \mathcal{I}(X', Y')$.

We look for criteria for membership in $\mathcal{I}$.

Examples of ideals

- $\mathcal{K}$ = compact operators.
- $\mathcal{WK}$ = weakly compact operators.
- $\mathcal{SS}$ = strictly singular operators.
- $\mathcal{DP}$ = Dunford-Pettis (completely continuous) operators: move weakly compact sets to relatively compact sets.
- $\mathcal{FSS}$ = finitely strictly singular operators. $T \in \mathcal{FSS}(X, Y)$ if $\forall \varepsilon > 0 \; \exists \, n \in \mathbb{N}$ so that any $E \subset X$ with $\dim E \geqslant n$ contains x with $\|Tx\| < \varepsilon \|x\|$.
- $\mathcal{IN}$ = inessential (or Fredholm Perturbation) operators: $T \in \mathcal{IN}(X, Y)$ if $I + UT$ is Fredholm for any $U \in B(Y, X)$. Known: suppose $B(X, Y)$ contains Fredholm operators. Then $T \in \mathcal{IN}(X, Y)$ iff $U + T$ is Fredholm for any Fredholm $U \in B(X, Y)$.

Examples of ideals

- $\mathcal{K}$ = compact operators.
- $\mathcal{WK}$ = weakly compact operators.
- $\mathcal{SS}$ = strictly singular operators.
- $\mathcal{DP}$ = Dunford-Pettis (completely continuous) operators: move weakly compact sets to relatively compact sets.
- $\mathcal{FSS}$ = finitely strictly singular operators. $T \in \mathcal{FSS}(X, Y)$ if $\forall \varepsilon > 0 \; \exists \, n \in \mathbb{N}$ so that any $E \subset X$ with $\dim E \geqslant n$ contains x with $\|Tx\| < \varepsilon \|x\|$.
- $\mathcal{IN}$ = inessential (or Fredholm Perturbation) operators: $T \in \mathcal{IN}(X, Y)$ if $I + UT$ is Fredholm for any $U \in B(Y, X)$. Known: suppose $B(X, Y)$ contains Fredholm operators. Then $T \in \mathcal{IN}(X, Y)$ iff $U + T$ is Fredholm for any Fredholm $U \in B(X, Y)$.

Examples of ideals

- $\mathcal{K}$ = compact operators.
- $\mathcal{WK}$ = weakly compact operators.
- $\mathcal{SS}$ = strictly singular operators.
- $\mathcal{DP}$ = Dunford-Pettis (completely continuous) operators: move weakly compact sets to relatively compact sets.
- $\mathcal{FSS}$ = finitely strictly singular operators. $T \in \mathcal{FSS}(X, Y)$ if $\forall \varepsilon > 0 \; \exists \, n \in \mathbb{N}$ so that any $E \subset X$ with $\dim E \geqslant n$ contains x with $\|Tx\| < \varepsilon \|x\|$.
- $\mathcal{IN}$ = inessential (or Fredholm Perturbation) operators: $T \in \mathcal{IN}(X, Y)$ if $I + UT$ is Fredholm for any $U \in B(Y, X)$. Known: suppose $B(X, Y)$ contains Fredholm operators. Then $T \in \mathcal{IN}(X, Y)$ iff $U + T$ is Fredholm for any Fredholm $U \in B(X, Y)$.

Examples of ideals

- $\mathcal{K}$ = compact operators.
- $\mathcal{WK}$ = weakly compact operators.
- $\mathcal{SS}$ = strictly singular operators.
- $\mathcal{DP}$ = Dunford-Pettis (completely continuous) operators: move weakly compact sets to relatively compact sets.
- $\mathcal{FSS}$ = finitely strictly singular operators. $T \in \mathcal{FSS}(X, Y)$ if $\forall \varepsilon > 0 \ \exists \, n \in \mathbb{N}$ so that any $E \subset X$ with $\dim E \geqslant n$ contains x with $\|Tx\| < \varepsilon \|x\|$.
- $\mathcal{IN}$ = inessential (or Fredholm Perturbation) operators: $T \in \mathcal{IN}(X, Y)$ if $I + UT$ is Fredholm for any $U \in B(Y, X)$. Known: suppose $B(X, Y)$ contains Fredholm operators. Then $T \in \mathcal{IN}(X, Y)$ iff $U + T$ is Fredholm for any Fredholm $U \in B(X, Y)$.

Examples of ideals

Proposition

For any Banach spaces X and Y,

$$\mathcal{K}(X, Y) \subseteq \mathcal{WK}(X, Y),$$

and

$$\mathcal{K}(X, Y) \subseteq \mathcal{FSS}(X, Y) \subseteq \mathcal{SS}(X, Y) \subseteq \mathcal{IN}(X, Y).$$

In general, these inclusions are proper, and there are no other inclusions.

Ordered spaces: definitions

A real Banach space E is an ordered Banach space (OBS) if it has a closed proper positive cone Z_+ ($Z_+ \cap (-Z_+) = \{0\}$).

The positive cone of an OBS Z is generating if $Z_+ - Z_+ = Z$. Equivalently, $\exists$ $\mathbf{G}_Z$ (the generating constant of Z) so that, $\forall$ $z \in Z$, $\exists$ $a, b \in Z_+$ s.t. $z = a - b$, and $\max\{\|a\|, \|b\|\} \leqslant \mathbf{G}_Z \|z\|$. Abusing the notation slightly, we call such OBSs generating.

An OBS Z is normal if there exists $\mathbf{N}_Z$ (the normality constant of Z) so that $\|z\| \leqslant \mathbf{N}_Z(\|a\| + \|b\|)$ whenever $a \leqslant z \leqslant b$.

Z is normal iff its dual $Z^\star$ is generating, and vice versa.

A complex OBS Z is the complexification of its real part $Z_{\mathbb{R}}$ ($Z = Z_{\mathbb{R}} + iZ_{\mathbb{R}}$). A complex OBS is normal (generating) if $Z_{\mathbb{R}}$ is. $\mathbf{G}_Z$, $\mathbf{N}_Z$ refer to $Z_{\mathbb{R}}$.

Ordered spaces: definitions

A real Banach space E is an ordered Banach space (OBS) if it has a closed proper positive cone Z_+ $(Z_+ \cap (-Z_+) = \{0\})$.

The positive cone of an OBS Z is generating if $Z_+ - Z_+ = Z$. Equivalently, $\exists\ \mathbf{G}_Z$ (the generating constant of Z) so that, $\forall$ $z \in Z$, $\exists\ a, b \in Z_+$ s.t. $z = a - b$, and $\max\{\|a\|, \|b\|\} \leqslant \mathbf{G}_Z \|z\|$. Abusing the notation slightly, we call such OBSs generating.

An OBS Z is normal if there exists $\mathbf{N}_Z$ (the normality constant of Z) so that $\|z\| \leqslant \mathbf{N}_Z(\|a\| + \|b\|)$ whenever $a \leqslant z \leqslant b$.

Z is normal iff its dual $Z^\star$ is generating, and vice versa.

A complex OBS Z is the complexification of its real part $Z_\mathbb{R}$ $(Z = Z_\mathbb{R} + iZ_\mathbb{R})$. A complex OBS is normal (generating) if $Z_\mathbb{R}$ is. $\mathbf{G}_Z$, $\mathbf{N}_Z$ refer to $Z_\mathbb{R}$.

Ordered spaces: definitions

A real Banach space E is an ordered Banach space (OBS) if it has a closed proper positive cone Z_+ $(Z_+ \cap (-Z_+) = \{0\})$.

The positive cone of an OBS Z is generating if $Z_+ - Z_+ = Z$. Equivalently, $\exists\ \mathbf{G}_Z$ (the generating constant of Z) so that, $\forall$ $z \in Z$, $\exists\ a, b \in Z_+$ s.t. $z = a - b$, and $\max\{\|a\|, \|b\|\} \leqslant \mathbf{G}_Z \|z\|$. Abusing the notation slightly, we call such OBSs generating.

An OBS Z is normal if there exists $\mathbf{N}_Z$ (the normality constant of Z) so that $\|z\| \leqslant \mathbf{N}_Z(\|a\| + \|b\|)$ whenever $a \leqslant z \leqslant b$.

Z is normal iff its dual $Z^\star$ is generating, and vice versa.

A complex OBS Z is the complexification of its real part $Z_\mathbb{R}$ $(Z = Z_\mathbb{R} + iZ_\mathbb{R})$. A complex OBS is normal (generating) if $Z_\mathbb{R}$ is. $\mathbf{G}_Z$, $\mathbf{N}_Z$ refer to $Z_\mathbb{R}$.

Ordered spaces: definitions

A real Banach space E is an ordered Banach space (OBS) if it has a closed proper positive cone Z_+ $(Z_+ \cap (-Z_+) = \{0\})$.

The positive cone of an OBS Z is generating if $Z_+ - Z_+ = Z$. Equivalently, $\exists$ $\mathbf{G}_Z$ (the generating constant of Z) so that, $\forall$ $z \in Z$, $\exists$ $a, b \in Z_+$ s.t. $z = a - b$, and $\max\{\|a\|, \|b\|\} \leqslant \mathbf{G}_Z\|z\|$. Abusing the notation slightly, we call such OBSs generating.

An OBS Z is normal if there exists $\mathbf{N}_Z$ (the normality constant of Z) so that $\|z\| \leqslant \mathbf{N}_Z(\|a\| + \|b\|)$ whenever $a \leqslant z \leqslant b$.

Z is normal iff its dual $Z^\star$ is generating, and vice versa.

A complex OBS Z is the complexification of its real part $Z_\mathbb{R}$ $(Z = Z_\mathbb{R} + iZ_\mathbb{R})$. A complex OBS is normal (generating) if $Z_\mathbb{R}$ is. $\mathbf{G}_Z$, $\mathbf{N}_Z$ refer to $Z_\mathbb{R}$.

Ordered spaces: definitions

A real Banach space E is an ordered Banach space (OBS) if it has a closed proper positive cone Z_+ ($Z_+ \cap (-Z_+) = \{0\}$).

The positive cone of an OBS Z is generating if $Z_+ - Z_+ = Z$. Equivalently, $\exists$ $\mathbf{G}_Z$ (the generating constant of Z) so that, $\forall$ $z \in Z$, $\exists$ $a, b \in Z_+$ s.t. $z = a - b$, and $\max\{\|a\|, \|b\|\} \leqslant \mathbf{G}_Z \|z\|$. Abusing the notation slightly, we call such OBSs generating.

An OBS Z is normal if there exists $\mathbf{N}_Z$ (the normality constant of Z) so that $\|z\| \leqslant \mathbf{N}_Z(\|a\| + \|b\|)$ whenever $a \leqslant z \leqslant b$.

Z is normal iff its dual $Z^\star$ is generating, and vice versa.

A complex OBS Z is the complexification of its real part $Z_\mathbb{R}$ ($Z = Z_\mathbb{R} + iZ_\mathbb{R}$). A complex OBS is normal (generating) if $Z_\mathbb{R}$ is. $\mathbf{G}_Z$, $\mathbf{N}_Z$ refer to $Z_\mathbb{R}$.

Problem I: domination

Question

Suppose $\mathcal{I}$ is an operator ideal, and X and Y are ordered Banach spaces. Suppose $T, S \in B(X, Y)$, $0 \leqslant T \leqslant S$, and $S \in \mathcal{I}(X, Y)$. Can we conclude $T \in \mathcal{I}(X, Y)$?

Problem II: coincidence and inclusion of ideals

Question

Suppose $\mathcal{I}$ and $\mathcal{J}$ are operator ideals, and X and Y are Banach spaces. Is it true that $\mathcal{I}(X,Y) \subset \mathcal{J}(X,Y)$? $\mathcal{I}(X,Y) = \mathcal{J}(X,Y)$?

Question

Suppose $\mathcal{I}$ and $\mathcal{J}$ are Banach operator ideals, and X and Y are ordered Banach spaces. Is it true that $\mathcal{I}(X,Y)_+ \subset \mathcal{J}(X,Y)_+$? $\mathcal{I}(X,Y)_+ = \mathcal{J}(X,Y)_+$?

Problem II: coincidence and inclusion of ideals

Question

Suppose $\mathcal{I}$ and $\mathcal{J}$ are operator ideals, and X and Y are Banach spaces. Is it true that $\mathcal{I}(X, Y) \subset \mathcal{J}(X, Y)$? $\mathcal{I}(X, Y) = \mathcal{J}(X, Y)$?

Question

Suppose $\mathcal{I}$ and $\mathcal{J}$ are Banach operator ideals, and X and Y are ordered Banach spaces. Is it true that $\mathcal{I}(X, Y)_+ \subset \mathcal{J}(X, Y)_+$? $\mathcal{I}(X, Y)_+ = \mathcal{J}(X, Y)_+$?

Reminder: domination in Banach lattices

Theorem (Aliprantis & Burkinshaw)

If E is a Banach lattice, $0 \leqslant T \leqslant S : E \to E$, and S is compact, then T^3 is compact. T^2 need not be compact.

Theorem (Fremlin & Dodds; Wickstead)

Suppose E and F are Banach lattices. TFAE:

1. *If $0 \leqslant T \leqslant S : E \to F$, and S is compact, then T is compact.*
2. *One of the three (non-exclusive) statements holds:*
   (i) *Both $E^\star$ and F are order continuous.*
   (ii) *F is atomic, and order continuous.*
   (iii) *$E^\star$ is atomic, and order continuous.*

Reminder: domination in Banach lattices

Theorem (Aliprantis & Burkinshaw)

If E is a Banach lattice, $0 \leqslant T \leqslant S : E \to E$, and S is compact, then T^3 is compact. T^2 need not be compact.

Theorem (Fremlin & Dodds; Wickstead)

Suppose E and F are Banach lattices. TFAE:

1. *If $0 \leqslant T \leqslant S : E \to F$, and S is compact, then T is compact.*
2. *One of the three (non-exclusive) statements holds:*
   - (i) *Both $E^\star$ and F are order continuous.*
   - (ii) *F is atomic, and order continuous.*
   - (iii) *$E^\star$ is atomic, and order continuous.*

Compact C^*-algebras: definitions

An element a of a Banach algebra $\mathcal{A}$ is multiplication compact if the map $\mathcal{A} \to \mathcal{A} : b \mapsto aba$ is compact.

A Banach algebra is compact if each of its elements is multiplication compact.

A C^*-algebra is compact iff it is C^*-isomorphic to $(\bigoplus_{i \in I} K(H_i))_{c_0}$.

Compact C^*-algebras: definitions

An element a of a Banach algebra $\mathcal{A}$ is multiplication compact if the map $\mathcal{A} \to \mathcal{A} : b \mapsto aba$ is compact.

A Banach algebra is compact if each of its elements is multiplication compact.

A C^*-algebra is compact iff it is C^*-isomorphic to $(\oplus_{i \in I} K(H_i))_{c_0}$.

Compact C^*-algebras: definitions

An element a of a Banach algebra $\mathcal{A}$ is multiplication compact if the map $\mathcal{A} \to \mathcal{A} : b \mapsto aba$ is compact.

A Banach algebra is compact if each of its elements is multiplication compact.

A C^*-algebra is compact iff it is C^*-isomorphic to $(\oplus_{i \in I} K(H_i))_{c_0}$.

Compact maps on compact C^*-algebras

Proposition

Suppose $\mathcal{A}$ is a compact C^*-algebra, E is a generating OBS, and $0 \leqslant T \leqslant S : E \to \mathcal{A}$. If S is compact, then so is T.

Proposition

For a C^*-algebra $\mathcal{A}$, TFAE:

1. *$\mathcal{A}$ is compact.*
2. *For any $c \in \mathcal{A}_+$, the order interval $[0, c] = \{a \in \mathcal{A} : 0 \leqslant a \leqslant c\}$ is compact.*
3. *For any $c \in \mathcal{A}_+$, the order interval $[0, c]$ is weakly compact.*
4. *$\mathcal{A}$ is a hereditary subalgebra of $\mathcal{A}^{**}$ (if $a \in \mathcal{A}, b \in \mathcal{A}^{**}, 0 \leqslant b \leqslant a$, then $b \in \mathcal{A}$).*

Compact maps on compact C^*-algebras

Proposition

Suppose $\mathcal{A}$ is a compact C^*-algebra, E is a generating OBS, and $0 \leqslant T \leqslant S : E \to \mathcal{A}$. If S is compact, then so is T.

Proposition

For a C^*-algebra $\mathcal{A}$, TFAE:

1. *$\mathcal{A}$ is compact.*
2. *For any $c \in \mathcal{A}_+$, the order interval $[0, c] = \{a \in \mathcal{A} : 0 \leqslant a \leqslant c\}$ is compact.*
3. *For any $c \in \mathcal{A}_+$, the order interval $[0, c]$ is weakly compact.*
4. *$\mathcal{A}$ is a hereditary subalgebra of $\mathcal{A}^{\star\star}$ (if $a \in \mathcal{A}, b \in \mathcal{A}^{\star\star}, 0 \leqslant b \leqslant a$, then $b \in \mathcal{A}$).*

Compact maps on scattered C^*-algebras

We say that a C^*-algebra $\mathcal{A}$ is scattered if the spectrum of any self-adjoint element of $\mathcal{A}$ is countable (equivalently, $\mathcal{A}^{\star\star} = (\sum_{i \in I} B(H_i))_\infty$).

Proposition

Suppose $\mathcal{A}$ is a scattered C^*-algebra, and E is a generating OBS. If $0 \leqslant T \leqslant S : E \to \mathcal{A}^\star$, and S is compact, then so is T.

Corollary

Suppose $\mathcal{A}$ is a scattered C^*-algebra, and E is a normal OBS. If $0 \leqslant T \leqslant S : \mathcal{A} \to E$, and S is compact, then so is T.

Compact maps on scattered C^*-algebras

We say that a C^*-algebra $\mathcal{A}$ is scattered if the spectrum of any self-adjoint element of $\mathcal{A}$ is countable (equivalently, $\mathcal{A}^{\star\star} = (\sum_{i \in I} B(H_i))_\infty$).

Proposition

Suppose $\mathcal{A}$ is a scattered C^*-algebra, and E is a generating OBS. If $0 \leqslant T \leqslant S : E \to \mathcal{A}^\star$, and S is compact, then so is T.

Corollary

Suppose $\mathcal{A}$ is a scattered C^*-algebra, and E is a normal OBS. If $0 \leqslant T \leqslant S : \mathcal{A} \to E$, and S is compact, then so is T.

Compact maps on scattered C^*-algebras

We say that a C^*-algebra $\mathcal{A}$ is scattered if the spectrum of any self-adjoint element of $\mathcal{A}$ is countable (equivalently, $\mathcal{A}^{\star\star} = (\sum_{i \in I} B(H_i))_\infty$).

Proposition

Suppose $\mathcal{A}$ is a scattered C^*-algebra, and E is a generating OBS. If $0 \leqslant T \leqslant S : E \to \mathcal{A}^\star$, and S is compact, then so is T.

Corollary

Suppose $\mathcal{A}$ is a scattered C^*-algebra, and E is a normal OBS. If $0 \leqslant T \leqslant S : \mathcal{A} \to E$, and S is compact, then so is T.

Compactness of operators on C^*-algebras

Theorem

Suppose $\mathcal{A}$ and $\mathcal{B}$ are C^*-algebras.

1. *Suppose at least one of the two conditions holds:*
   (i) *$\mathcal{A}$ is scattered.*
   (ii) *$\mathcal{B}$ is compact.*

   If $0 \leqslant T \leqslant S : \mathcal{A} \to \mathcal{B}$, and S is compact, then T is compact.
2. *Suppose $\mathcal{A}$ is not scattered, and $\mathcal{B}$ is not compact. Then there exist $0 \leqslant T \leqslant S : \mathcal{A} \to \mathcal{B}$, so that S has rank 1, while T is not compact.*

Compactness of operators on C^*-algebras

Theorem

Suppose $\mathcal{A}$ and $\mathcal{B}$ are C^*-algebras.

1. *Suppose at least one of the two conditions holds:*
   - (i) *$\mathcal{A}$ is scattered.*
   - (ii) *$\mathcal{B}$ is compact.*

   If $0 \leqslant T \leqslant S : \mathcal{A} \to \mathcal{B}$, and S is compact, then T is compact.
2. *Suppose $\mathcal{A}$ is not scattered, and $\mathcal{B}$ is not compact. Then there exist $0 \leqslant T \leqslant S : \mathcal{A} \to \mathcal{B}$, so that S has rank 1, while T is not compact.*

Compactness of multiplication operators on C^*-algebras

Suppose $\mathcal{A} \subset B(H)$. For $x \in B(H)$, define
$\mathbf{M}_x : \mathcal{A} \to B(H) : a \mapsto x^*ax$.

Proposition

Suppose x is an element of a C^*-algebra $\mathcal{A}$.

1. *If $\mathbf{M}_x$ is weakly compact, and $0 \leqslant T \leqslant \mathbf{M}_x : \mathcal{A} \to \mathcal{A}$, then T is compact.*
2. *If $0 \leqslant \mathbf{M}_x \leqslant S : \mathcal{A} \to \mathcal{A}$, and S is weakly compact, then $\mathbf{M}_x$ is compact.*

Consequently, $\mathbf{M}_x$ is compact iff it is weakly compact.

Compactness of multiplication operators on C^*-algebras

Suppose $\mathcal{A} \subset B(H)$. For $x \in B(H)$, define
$\mathbf{M}_x : \mathcal{A} \to B(H) : a \mapsto x^* a x$.

Proposition

Suppose x is an element of a C^*-algebra $\mathcal{A}$.

1. *If $\mathbf{M}_x$ is weakly compact, and $0 \leqslant T \leqslant \mathbf{M}_x : \mathcal{A} \to \mathcal{A}$, then T is compact.*
2. *If $0 \leqslant \mathbf{M}_x \leqslant S : \mathcal{A} \to \mathcal{A}$, and S is weakly compact, then $\mathbf{M}_x$ is compact.*

Consequently, $\mathbf{M}_x$ is compact iff it is weakly compact.

Compactness of multiplication operators on C^*-algebras

Suppose $\mathcal{A} \subset B(H)$. For $x \in B(H)$, define
$\mathbf{M}_x : \mathcal{A} \to B(H) : a \mapsto x^* a x$.

Proposition

Suppose x is an element of a C^*-algebra $\mathcal{A}$.

1. *If $\mathbf{M}_x$ is weakly compact, and $0 \leqslant T \leqslant \mathbf{M}_x : \mathcal{A} \to \mathcal{A}$, then T is compact.*
2. *If $0 \leqslant \mathbf{M}_x \leqslant S : \mathcal{A} \to \mathcal{A}$, and S is weakly compact, then $\mathbf{M}_x$ is compact.*

Consequently, $\mathbf{M}_x$ is compact iff it is weakly compact.

Compactness of multiplication operators on C^*-algebras

Suppose $\mathcal{A} \subset B(H)$. For $x \in B(H)$, define
$\mathbf{M}_x : \mathcal{A} \to B(H) : a \mapsto x^* a x$.

Proposition

Suppose x is an element of a C^*-algebra $\mathcal{A}$.

1. *If $\mathbf{M}_x$ is weakly compact, and $0 \leqslant T \leqslant \mathbf{M}_x : \mathcal{A} \to \mathcal{A}$, then T is compact.*
2. *If $0 \leqslant \mathbf{M}_x \leqslant S : \mathcal{A} \to \mathcal{A}$, and S is weakly compact, then $\mathbf{M}_x$ is compact.*

Consequently, $\mathbf{M}_x$ is compact iff it is weakly compact.

Compactness of multiplication operators on C^*-algebras: the irreducible case

Proposition

Suppose $\mathcal{A}$ is an irreducible C^*-subalgebra of $B(H)$, and $x \in B(H)$.

1. *If $\mathbf{M}_x : \mathcal{A} \to B(H)$ is compact, and $0 \leqslant T \leqslant \mathbf{M}_x$, then T is compact.*
2. *If $S : \mathcal{A} \to B(H)$ is compact, and $0 \leqslant \mathbf{M}_x \leqslant S$, then $\mathbf{M}_x$ is compact.*

Remark

Irreducibility is essential: there exists an abelian C^*-subalgebra $\mathcal{A} \subset B(H)$, and $x, y \in B(H)$, so that $0 \leqslant \mathbf{M}_x \leqslant \mathbf{M}_y : \mathcal{A} \to B(H)$, $\mathbf{M}_y$ is compact, while $\mathbf{M}_x$ is not.

Compactness of multiplication operators on C^*-algebras: the irreducible case

Proposition

Suppose $\mathcal{A}$ is an irreducible C^*-subalgebra of $B(H)$, and $x \in B(H)$.

1. *If $\mathbf{M}_x : \mathcal{A} \to B(H)$ is compact, and $0 \leqslant T \leqslant \mathbf{M}_x$, then T is compact.*
2. *If $S : \mathcal{A} \to B(H)$ is compact, and $0 \leqslant \mathbf{M}_x \leqslant S$, then $\mathbf{M}_x$ is compact.*

Remark

Irreducibility is essential: there exists an abelian C^*-subalgebra $\mathcal{A} \subset B(H)$, and $x, y \in B(H)$, so that $0 \leqslant \mathbf{M}_x \leqslant \mathbf{M}_y : \mathcal{A} \to B(H)$, $\mathbf{M}_y$ is compact, while $\mathbf{M}_x$ is not.

Compactness of multiplication operators on C^*-algebras: the irreducible case

Proposition

Suppose $\mathcal{A}$ is an irreducible C^*-subalgebra of $B(H)$, and $x \in B(H)$.

1. *If $\mathbf{M}_x : \mathcal{A} \to B(H)$ is compact, and $0 \leqslant T \leqslant \mathbf{M}_x$, then T is compact.*
2. *If $S : \mathcal{A} \to B(H)$ is compact, and $0 \leqslant \mathbf{M}_x \leqslant S$, then $\mathbf{M}_x$ is compact.*

Remark

Irreducibility is essential: there exists an abelian C^*-subalgebra $\mathcal{A} \subset B(H)$, and $x, y \in B(H)$, so that $0 \leqslant \mathbf{M}_x \leqslant \mathbf{M}_y : \mathcal{A} \to B(H)$, $\mathbf{M}_y$ is compact, while $\mathbf{M}_x$ is not.

Weakly compact operators on Schatten spaces

A symmetric sequence space $\mathcal{E}$ gives rise to a Schatten space $\mathfrak{C}_{\mathcal{E}}(H) = \{T \in K(H) : (s_i(T)) \in \mathcal{E}\}$, with $\|T\|_{\mathcal{E}} = \|(s_i(T))_{i \in \mathbb{N}}\|_{\mathcal{E}}$.
Convention: $\mathfrak{C}_p = \mathfrak{C}_{\ell_p}$.

Theorem

Suppose $\mathcal{E}$ is a separable symmetric sequence space, containing ℓ_1, F is a Banach lattice, and H is an inf. dim. Hilbert space. TFAE:

(1) F is order continuous.

(2) If $0 \leqslant T \leqslant S : \mathfrak{C}_{\mathcal{E}}(H) \to F$, and S is weakly compact, then T is weakly compact as well.

Weakly compact operators on Schatten spaces

A symmetric sequence space $\mathcal{E}$ gives rise to a Schatten space $\mathfrak{C}_{\mathcal{E}}(H) = \{T \in K(H) : (s_i(T)) \in \mathcal{E}\}$, with $\|T\|_{\mathcal{E}} = \|(s_i(T))_{i\in\mathbb{N}}\|_{\mathcal{E}}$.
Convention: $\mathfrak{C}_p = \mathfrak{C}_{\ell_p}$.

Theorem

Suppose $\mathcal{E}$ is a separable symmetric sequence space, containing ℓ_1, F is a Banach lattice, and H is an inf. dim. Hilbert space. TFAE:

(1) *F is order continuous.*

(2) *If $0 \leqslant T \leqslant S : \mathfrak{C}_{\mathcal{E}}(H) \to F$, and S is weakly compact, then T is weakly compact as well.*

Dunford-Pettis Schur multipliers

An operator $T \in B(E,F)$ is Dunford-Pettis if $\lim_n \|Tx_n\| = 0$ whenever $x_n \xrightarrow[weakly]{} 0$. Equivalently, the image of any weakly compact set is compact.

Theorem

Suppose $0 \leqslant \mathbf{S}_\phi \leqslant \mathbf{S}_\psi$ are Schur multipliers from $\mathfrak{C}_1$ to $\mathfrak{C}_\mathcal{E}$ ($\mathcal{E}$ is a symmetric sequence space). If $\mathbf{S}_\psi$ is Dunford-Pettis, then the same is true for $\mathbf{S}_\phi$.

There exist examples of Banach lattices E, F, and $0 \leqslant T \leqslant S : E \to F$, so that S is Dunford-Pettis, but T is not.

Dunford-Pettis Schur multipliers

An operator $T \in B(E, F)$ is Dunford-Pettis if $\lim_n \|Tx_n\| = 0$ whenever $x_n \xrightarrow[weakly]{} 0$. Equivalently, the image of any weakly compact set is compact.

Theorem

Suppose $0 \leqslant \mathbf{S}_\phi \leqslant \mathbf{S}_\psi$ are Schur multipliers from $\mathfrak{C}_1$ to $\mathfrak{C}_\mathcal{E}$ ($\mathcal{E}$ is a symmetric sequence space). If $\mathbf{S}_\psi$ is Dunford-Pettis, then the same is true for $\mathbf{S}_\phi$.

There exist examples of Banach lattices E, F, and $0 \leqslant T \leqslant S : E \to F$, so that S is Dunford-Pettis, but T is not.

Dunford-Pettis Schur multipliers

An operator $T \in B(E, F)$ is Dunford-Pettis if $\lim_n \|Tx_n\| = 0$ whenever $x_n \xrightarrow[weakly]{} 0$. Equivalently, the image of any weakly compact set is compact.

Theorem

Suppose $0 \leqslant \mathbf{S}_\phi \leqslant \mathbf{S}_\psi$ are Schur multipliers from $\mathfrak{C}_1$ to $\mathfrak{C}_\mathcal{E}$ ($\mathcal{E}$ is a symmetric sequence space). If $\mathbf{S}_\psi$ is Dunford-Pettis, then the same is true for $\mathbf{S}_\phi$.

There exist examples of Banach lattices E, F, and $0 \leqslant T \leqslant S : E \to F$, so that S is Dunford-Pettis, but T is not.

Part II: coincidence and inclusion of ideals

Question

Suppose $\mathcal{I}$ and $\mathcal{J}$ are operator ideals, and X and Y are Banach spaces. Is it true that one of the following holds:

- $\mathcal{I}(X, Y) \subset \mathcal{J}(X, Y)$;
- $\mathcal{I}(X, Y) = \mathcal{J}(X, Y)$;
- $\mathcal{I}(X, Y)_+ \subset \mathcal{J}(X, Y)_+$;
- $\mathcal{I}(X, Y)_+ = \mathcal{J}(X, Y)_+$.

Early results

Theorem (Calkin 1940; Gohberg, Feldman, and Markus 1960)

$\mathcal{K}(\ell_p) = \mathcal{FSS}(\ell_p) = \mathcal{SS}(\ell_p) = \mathcal{IN}(\ell_p) = \mathcal{DP}(\ell_p)$ *(*$1 \leqslant p < \infty$*),*
same for c_0*.*

Theorem (Pelczynski 1965)

$\mathcal{WK}(C(K)) = \mathcal{FSS}(C(K)) = \mathcal{SS}(C(K))$.
Moreover, for $T \in B(C(K))$*, TFAE:*

1. $T \notin \mathcal{WK}(C(K)) = \mathcal{FSS}(C(K)) = \mathcal{SS}(C(K))$.
2. *There exists a subspace* $E \subset C(K)$*, isomorphic to* c_0*, so that* $T|_E$ *is an isomorphism (*T *fixes* c_0*).*

Similar results hold for L_1*.*

Early results

Theorem (Calkin 1940; Gohberg, Feldman, and Markus 1960)

$\mathcal{K}(\ell_p) = \mathcal{FSS}(\ell_p) = \mathcal{SS}(\ell_p) = \mathcal{IN}(\ell_p) = \mathcal{DP}(\ell_p)$ $(1 \leqslant p < \infty)$, same for c_0.

Theorem (Pelczynski 1965)

$\mathcal{WK}(C(K)) = \mathcal{FSS}(C(K)) = \mathcal{SS}(C(K))$.

Moreover, for $T \in B(C(K))$, TFAE:

1. $T \notin \mathcal{WK}(C(K)) = \mathcal{FSS}(C(K)) = \mathcal{SS}(C(K))$.
2. *There exists a subspace $E \subset C(K)$, isomorphic to c_0, so that $T|_E$ is an isomorphism (T fixes c_0).*

Similar results hold for L_1.

Early results

Theorem (Calkin 1940; Gohberg, Feldman, and Markus 1960)

$\mathcal{K}(\ell_p) = \mathcal{FSS}(\ell_p) = \mathcal{SS}(\ell_p) = \mathcal{IN}(\ell_p) = \mathcal{DP}(\ell_p)$ $(1 \leqslant p < \infty)$, same for c_0.

Theorem (Pelczynski 1965)

$\mathcal{WK}(C(K)) = \mathcal{FSS}(C(K)) = \mathcal{SS}(C(K))$.
Moreover, for $T \in B(C(K))$, TFAE:

1. $T \notin \mathcal{WK}(C(K)) = \mathcal{FSS}(C(K)) = \mathcal{SS}(C(K))$.
2. *There exists a subspace $E \subset C(K)$, isomorphic to c_0, so that $T|_E$ is an isomorphism (T fixes c_0).*

Similar results hold for L_1.

Early results

Theorem (Calkin 1940; Gohberg, Feldman, and Markus 1960)

$\mathcal{K}(\ell_p) = \mathcal{FSS}(\ell_p) = \mathcal{SS}(\ell_p) = \mathcal{IN}(\ell_p) = \mathcal{DP}(\ell_p)$ *($1 \leqslant p < \infty$), same for c_0.*

Theorem (Pelczynski 1965)

$\mathcal{WK}(C(K)) = \mathcal{FSS}(C(K)) = \mathcal{SS}(C(K))$.
Moreover, for $T \in B(C(K))$, TFAE:

1. $T \notin \mathcal{WK}(C(K)) = \mathcal{FSS}(C(K)) = \mathcal{SS}(C(K))$.
2. *There exists a subspace $E \subset C(K)$, isomorphic to c_0, so that $T|_E$ is an isomorphism (T fixes c_0).*

Similar results hold for L_1.

More recent results

[L. Weis 1977]: description of strictly singular operators on $L_p(0,1)$.

[G. Russu 1984, P. Tradacete 2010]: description of strictly singular operators on Lorentz spaces.

Theorem (Lefevre 2012)

$\mathcal{WK}(X) = \mathcal{FSS}(X) = \mathcal{SS}(X)$ if X is the disc algebra, or a subspace of $C(K)$ with reflexive annihilator.

More recent results

[L. Weis 1977]: description of strictly singular operators on $L_p(0,1)$.

[G. Russu 1984, P. Tradacete 2010]: description of strictly singular operators on Lorentz spaces.

Theorem (Lefevre 2012)

$\mathcal{WK}(X) = \mathcal{FSS}(X) = \mathcal{SS}(X)$ if X is the disc algebra, or a subspace of $C(K)$ with reflexive annihilator.

Operators on C^*-algebras

Theorem

Suppose $\mathcal{A}$ is a C^*-algebra, and X is a Banach space. For $T \in B(\mathcal{A}, X)$, TFAE:

1. *T is not strictly singular.*
2. *T fixes either a copy of c_0, or a complemented copy of ℓ_2.*

If $\mathcal{A}$ is a von Neumann algebra, then (1) and (2) are equivalent to

3. *T fixes either a complemented copy of ℓ_2, or a copy of ℓ_∞, complemented by a weak* continuous projection. Furthermore, if X is a dual space, and T is weak* continuous, then a copy of ℓ_2 can be chosen so that it is complemented by a weak*-continuous projection.*

Operators on C^*-algebras

Theorem

Suppose $\mathcal{A}$ is a C^*-algebra, and X is a Banach space. For $T \in B(\mathcal{A}, X)$, TFAE:

1. *T is not strictly singular.*
2. *T fixes either a copy of c_0, or a complemented copy of ℓ_2.*

If $\mathcal{A}$ is a von Neumann algebra, then (1) and (2) are equivalent to

3. *T fixes either a complemented copy of ℓ_2, or a copy of ℓ_∞, complemented by a weak* continuous projection. Furthermore, if X is a dual space, and T is weak* continuous, then a copy of ℓ_2 can be chosen so that it is complemented by a weak*-continuous projection.*

Operators on C^*-algebras

Theorem

Suppose $\mathcal{A}$ is a C^*-algebra, and X is a Banach space. For $T \in B(\mathcal{A}, X)$, TFAE:

1. *T is not strictly singular.*
2. *T fixes either a copy of c_0, or a complemented copy of ℓ_2.*

If $\mathcal{A}$ is a von Neumann algebra, then (1) and (2) are equivalent to

3. *T fixes either a complemented copy of ℓ_2, or a copy of ℓ_∞, complemented by a weak* continuous projection.*
   Furthermore, if X is a dual space, and T is weak* continuous, then a copy of ℓ_2 can be chosen so that it is complemented by a weak*-continuous projection.

Operators on C^*-algebras

Theorem

Suppose $\mathcal{A}$ is a C^*-algebra, and X is a Banach space. For $T \in B(\mathcal{A}, X)$, TFAE:

1. *T is not strictly singular.*
2. *T fixes either a copy of c_0, or a complemented copy of ℓ_2.*

If $\mathcal{A}$ is a von Neumann algebra, then (1) and (2) are equivalent to

3. *T fixes either a complemented copy of ℓ_2, or a copy of ℓ_∞, complemented by a weak* continuous projection. Furthermore, if X is a dual space, and T is weak* continuous, then a copy of ℓ_2 can be chosen so that it is complemented by a weak*-continuous projection.*

Operators on C^*-algebras

A von Neumann algebra $\mathcal{A}$ is *finite type I* if it is a direct sum of finitely many type I_n algebras ($n \in \mathbb{N}$).

Proposition

A von Neumann algebra $\mathcal{A}$ is of finite type I if and only if $\mathcal{FSS}(\mathcal{A}) = \mathcal{SS}(A) = \mathcal{IN}(\mathcal{A}) = \mathcal{WK}(\mathcal{A})$. Moreover, if $\mathcal{A}$ is not of finite type I, then all of these classes are different.

Non-commutative sequence and L_p spaces

Schatten spaces. If $\mathcal{E}$ is a symmetric sequence space, then $\mathfrak{C}_{\mathcal{E}} = \{T \in \mathcal{K}(\ell_2) : (s_i(T))_{i\in\mathbb{N}} \in \mathcal{E}\}$, with the norm $\|T\|_{\mathfrak{C}_{\mathcal{E}}} = \|(s_i(T))_{i\in\mathbb{N}}\|_{\mathcal{E}}$.

Notation: $\mathfrak{C}_p = \mathfrak{C}_{\ell_p}$, for $1 \leqslant p < \infty$. $\mathfrak{C}_\infty = \mathfrak{C}_{c_0} = \mathcal{K}(\ell_2)$.

Lorentz sequence spaces. Suppose $w(1) \geqslant w(2) \geqslant \ldots > 0$ and $\sum_k w(k) = \infty$. Then $\mathfrak{l}(w,p)$ is the completion of c_{00} w.r.t. the norm $\|(x_i)\| = \left(\sum_k w(k) x_k^{\dagger p}\right)^{1/p}$ ($(x_k^\dagger)$ is the non-increasing rearrangement of $(|x_k|)$).

Non-commutative L_p. If τ is a normal faithful semifinite state on a von Neumann algebra $\mathcal{A}$, set $\mathcal{A}_0 = \{a \in \mathcal{A} : \tau(|a|) < \infty\}$. For $a \in \mathcal{A}_0$ set $\|a\|_p = \left(\tau(|a|^p)\right)^{1/p}$. $L_p(\tau)$ is the completion of $\mathcal{A}_0$ in $\|\cdot\|_p$.

Non-commutative sequence and L_p spaces

Schatten spaces. If $\mathcal{E}$ is a symmetric sequence space, then $\mathfrak{C}_{\mathcal{E}} = \{T \in \mathcal{K}(\ell_2) : (s_i(T))_{i\in\mathbb{N}} \in \mathcal{E}\}$, with the norm $\|T\|_{\mathfrak{C}_{\mathcal{E}}} = \|(s_i(T))_{i\in\mathbb{N}}\|_{\mathcal{E}}$.

Notation: $\mathfrak{C}_p = \mathfrak{C}_{\ell_p}$, for $1 \leqslant p < \infty$. $\mathfrak{C}_\infty = \mathfrak{C}_{c_0} = \mathcal{K}(\ell_2)$.

Lorentz sequence spaces. Suppose $w(1) \geqslant w(2) \geqslant \ldots > 0$ and $\sum_k w(k) = \infty$. Then $\mathfrak{l}(w,p)$ is the completion of c_{00} w.r.t. the norm $\|(x_i)\| = \left(\sum_k w(k) x_k^{\dagger p}\right)^{1/p}$ ($(x_k^\dagger)$ is the non-increasing rearrangement of $(|x_k|)$).

Non-commutative L_p. If τ is a normal faithful semifinite state on a von Neumann algebra $\mathcal{A}$, set $\mathcal{A}_0 = \{a \in \mathcal{A} : \tau(|a|) < \infty\}$. For $a \in \mathcal{A}_0$ set $\|a\|_p = \left(\tau(|a|^p)\right)^{1/p}$. $L_p(\tau)$ is the completion of $\mathcal{A}_0$ in $\|\cdot\|_p$.

Non-commutative sequence and L_p spaces

Schatten spaces. If $\mathcal{E}$ is a symmetric sequence space, then $\mathfrak{C}_{\mathcal{E}} = \{T \in \mathcal{K}(\ell_2) : (s_i(T))_{i\in\mathbb{N}} \in \mathcal{E}\}$, with the norm $\|T\|_{\mathfrak{C}_{\mathcal{E}}} = \|(s_i(T))_{i\in\mathbb{N}}\|_{\mathcal{E}}$.

Notation: $\mathfrak{C}_p = \mathfrak{C}_{\ell_p}$, for $1 \leqslant p < \infty$. $\mathfrak{C}_\infty = \mathfrak{C}_{c_0} = \mathcal{K}(\ell_2)$.

Lorentz sequence spaces. Suppose $w(1) \geqslant w(2) \geqslant \ldots > 0$ and $\sum_k w(k) = \infty$. Then $\mathfrak{l}(w,p)$ is the completion of c_{00} w.r.t. the norm $\|(x_i)\| = \left(\sum_k w(k) x_k^{\dagger p}\right)^{1/p}$ ($(x_k^\dagger)$ is the non-increasing rearrangement of $(|x_k|)$).

Non-commutative L_p. If τ is a normal faithful semifinite state on a von Neumann algebra $\mathcal{A}$, set $\mathcal{A}_0 = \{a \in \mathcal{A} : \tau(|a|) < \infty\}$. For $a \in \mathcal{A}_0$ set $\|a\|_p = \left(\tau(|a|^p)\right)^{1/p}$. $L_p(\tau)$ is the completion of $\mathcal{A}_0$ in $\|\cdot\|_p$.

Non-commutative sequence and L_p spaces

Schatten spaces. If $\mathcal{E}$ is a symmetric sequence space, then $\mathfrak{C}_{\mathcal{E}} = \{T \in \mathcal{K}(\ell_2) : (s_i(T))_{i\in\mathbb{N}} \in \mathcal{E}\}$, with the norm $\|T\|_{\mathfrak{C}_{\mathcal{E}}} = \|(s_i(T))_{i\in\mathbb{N}}\|_{\mathcal{E}}$.

Notation: $\mathfrak{C}_p = \mathfrak{C}_{\ell_p}$, for $1 \leqslant p < \infty$. $\mathfrak{C}_\infty = \mathfrak{C}_{c_0} = \mathcal{K}(\ell_2)$.

Lorentz sequence spaces. Suppose $w(1) \geqslant w(2) \geqslant \ldots > 0$ and $\sum_k w(k) = \infty$. Then $\mathfrak{l}(w,p)$ is the completion of c_{00} w.r.t. the norm $\|(x_i)\| = \left(\sum_k w(k) x_k^{\dagger p}\right)^{1/p}$ ($(x_k^\dagger)$ is the non-increasing rearrangement of $(|x_k|)$).

Non-commutative L_p. If τ is a normal faithful semifinite state on a von Neumann algebra $\mathcal{A}$, set $\mathcal{A}_0 = \{a \in \mathcal{A} : \tau(|a|) < \infty\}$. For $a \in \mathcal{A}_0$ set $\|a\|_p = \left(\tau(|a|^p)\right)^{1/p}$. $L_p(\tau)$ is the completion of $\mathcal{A}_0$ in $\|\cdot\|_p$.

L_p and Schatten spaces and their relatives

Theorem

Suppose that either

(i) $X = \mathfrak{C}_{\mathfrak{l}(p,w)}$ $(1 \leqslant p < \infty)$, *or*

(ii) $X = L_p(\tau)$ $(1 < p < \infty$, τ *is a normal faithful finite trace on a hyperfinite von Neumann algebra*$)$.

Then, for $T \in B(X)$, *TFAE:*

1. T *is not strictly singular.*
2. T *is not inessential.*
3. X *contains a subspace* E, *isomorphic to either* ℓ_2 *or* ℓ_p, *so that* $T|_E$ *is an isomorphism, and both* E *and* $T(E)$ *are complemented in* X.

More on Schatten and L_p spaces

Theorem

1. $\mathcal{SS}(\mathfrak{C}_p, \mathfrak{C}_q) = \mathcal{FSS}(\mathfrak{C}_p, \mathfrak{C}_q) = \mathcal{K}(\mathfrak{C}_p, \mathfrak{C}_q)$ *if* $\infty \geq p \geq 2 \geq q \geq 1$,
2. $\mathcal{SS}(\mathfrak{C}_p, \mathfrak{C}_q) \supsetneq \mathcal{FSS}(\mathfrak{C}_p, \mathfrak{C}_q) \supsetneq \mathcal{K}(\mathfrak{C}_p, \mathfrak{C}_q)$ *otherwise.*

Theorem

Suppose $p, q \in (1, \infty)$, $p \neq q$, *and* τ *is a normal faithful finite trace on a hyperfinite von Neumann algebra. For* $T \in B(L_p(\tau), L_q(\tau))$, *TFAE:*

1. T *is not strictly singular.*
2. T *is an isomorphism on* $E \subset L_p(\tau)$, *where* E *is isomorphic to* ℓ_2, *and both* E *and* $T(E)$ *are complemented.*

More on Schatten and L_p spaces

Theorem

1. $\mathcal{SS}(\mathfrak{C}_p, \mathfrak{C}_q) = \mathcal{FSS}(\mathfrak{C}_p, \mathfrak{C}_q) = \mathcal{K}(\mathfrak{C}_p, \mathfrak{C}_q)$ *if* $\infty \geq p \geq 2 \geq q \geq 1$,
2. $\mathcal{SS}(\mathfrak{C}_p, \mathfrak{C}_q) \supsetneq \mathcal{FSS}(\mathfrak{C}_p, \mathfrak{C}_q) \supsetneq \mathcal{K}(\mathfrak{C}_p, \mathfrak{C}_q)$ *otherwise.*

Theorem

Suppose $p, q \in (1, \infty)$, $p \neq q$, *and* τ *is a normal faithful finite trace on a hyperfinite von Neumann algebra. For* $T \in B(L_p(\tau), L_q(\tau))$, *TFAE:*

1. T *is not strictly singular.*
2. T *is an isomorphism on* $E \subset L_p(\tau)$, *where* E *is isomorphic to* ℓ_2, *and both* E *and* $T(E)$ *are complemented.*

Positive operators on Schatten spaces

Theorem

For $1 \leqslant q < p < \infty$, $B(\mathfrak{C}_p, \mathfrak{C}_q)_+ = \mathcal{K}(\mathfrak{C}_p, \mathfrak{C}_q)_+$.

Theorem

For $1 \leqslant p < \infty$, and $T \in B(\mathfrak{C}_p)_+$, TFAE:

1. *T is not compact.*
2. *T is not strictly singular.*
3. *There is a subspace $E \subset \mathfrak{C}_p$, isomorphic to ℓ_p, so that $T|_E$ is an isomorphism, and both E and $T(E)$ are complemented.*

Positive operators on Schatten spaces

Theorem

For $1 \leqslant q < p < \infty$, $B(\mathfrak{C}_p, \mathfrak{C}_q)_+ = \mathcal{K}(\mathfrak{C}_p, \mathfrak{C}_q)_+$.

Theorem

For $1 \leqslant p < \infty$, *and* $T \in B(\mathfrak{C}_p)_+$, *TFAE:*

1. T *is not compact.*
2. T *is not strictly singular.*
3. *There is a subspace* $E \subset \mathfrak{C}_p$, *isomorphic to* ℓ_p, *so that* $T|_E$ *is an isomorphism, and both* E *and* $T(E)$ *are complemented.*

Positive operators on L_p spaces

Proposition

Suppose τ is a normal faithful finite traces on a hyperfinite von Neumann algebra. Then $\mathcal{SS}(L_p(\tau), L_q(\tau))_+ = \mathcal{K}(L_p(\tau), L_q(\tau))_+$ iff $1 < q \leqslant p < \infty$. In particular, $\mathcal{SS}(L_p(\tau))_+ = \mathcal{K}(L_p(\tau))_+$ for $1 < p < \infty$.

Positive operators on von Neumann algebra preduals

Recall that the positive cone X_+ in an ordered real Banach space X is proper if $X_+ \cap (-X_+) = \{0\}$, and generating if $X = X_+ - X_+$.

Proposition

(1) Suppose X is an ordered Banach space with a proper generating cone, and a von Neumann algebra $\mathcal{A}$ is purely atomic. Then $\mathcal{IN}(X, \mathcal{A}_\star)_+ = \mathcal{SS}(X, \mathcal{A}_\star)_+ = \mathcal{WK}(X, \mathcal{A}_\star)_+ = \mathcal{K}(X, \mathcal{A}_\star)_+$.

(2) If $\mathcal{A}$ is not purely atomic, then $(\mathcal{IN}(\mathcal{A}_\star)_+ \cap \mathcal{SS}(\mathcal{A}_\star)_+ \cap \mathcal{WK}(\mathcal{A}_\star)_+) \backslash \mathcal{K}(\mathcal{A}_\star)_+ \neq \emptyset$.

Positive operators on von Neumann algebra preduals

Recall that the positive cone X_+ in an ordered real Banach space X is proper if $X_+ \cap (-X_+) = \{0\}$, and generating if $X = X_+ - X_+$.

Proposition

(1) Suppose X is an ordered Banach space with a proper generating cone, and a von Neumann algebra $\mathcal{A}$ is purely atomic. Then $\mathcal{IN}(X, \mathcal{A}_\star)_+ = \mathcal{SS}(X, \mathcal{A}_\star)_+ = \mathcal{WK}(X, \mathcal{A}_\star)_+ = \mathcal{K}(X, \mathcal{A}_\star)_+$.

(2) If $\mathcal{A}$ is not purely atomic, then $(\mathcal{IN}(\mathcal{A}_\star)_+ \cap \mathcal{SS}(\mathcal{A}_\star)_+ \cap \mathcal{WK}(\mathcal{A}_\star)_+) \backslash \mathcal{K}(\mathcal{A}_\star)_+ \neq \emptyset$.

Positive operators on von Neumann algebra preduals

Recall that the positive cone X_+ in an ordered real Banach space X is proper if $X_+ \cap (-X_+) = \{0\}$, and generating if $X = X_+ - X_+$.

Proposition

(1) Suppose X is an ordered Banach space with a proper generating cone, and a von Neumann algebra $\mathcal{A}$ is purely atomic. Then $\mathcal{IN}(X, \mathcal{A}_\star)_+ = \mathcal{SS}(X, \mathcal{A}_\star)_+ = \mathcal{WK}(X, \mathcal{A}_\star)_+ = \mathcal{K}(X, \mathcal{A}_\star)_+$.

(2) If $\mathcal{A}$ is not purely atomic, then $\left(\mathcal{IN}(\mathcal{A}_\star)_+ \cap \mathcal{SS}(\mathcal{A}_\star)_+ \cap \mathcal{WK}(\mathcal{A}_\star)_+\right) \backslash \mathcal{K}(\mathcal{A}_\star)_+ \neq \emptyset$.

Dunford-Pettis operators

Proposition

Suppose X is a Banach space. Then $T \in B(\mathfrak{C}_1, X)$ is Dunford-Pettis if and only if $T|_E$ is compact whenever E is isomorphic to ℓ_2.

Products of operators

Proposition

Suppose either $X = \mathfrak{C}_p$ $(1 < p \leqslant \infty)$, *or* $X = L_p(\tau)$ $(1 < p < \infty$, *and* τ *is a normal faithful finite trace on a hyperfinite von Neumann algebra*$)$. *If* $T, S \in \mathcal{SS}(X)$, *then* TS *is compact.*

Proposition

A von Neumann algebra is of finite type I if and only if the product of any two strictly singular operators on it is compact.

Products of operators

Proposition

Suppose either $X = \mathfrak{C}_p$ $(1 < p \leqslant \infty)$, *or* $X = L_p(\tau)$ $(1 < p < \infty$, *and* τ *is a normal faithful finite trace on a hyperfinite von Neumann algebra*). *If* $T, S \in \mathcal{SS}(X)$, *then* TS *is compact.*

Proposition

A von Neumann algebra is of finite type I if and only if the product of any two strictly singular operators on it is compact.

Tools used

- Kadec-Pelczynski style results on equiintegrability
- “Non-commutative gliding hump”
- Structure of Schatten spaces, due to Arazy.
- Subprojectivity.

Subprojective spaces

[R.Whitley 1964] A Banach space X is subprojective if, for any infinite dimensional $Y \subset X$, there exists an infinite dimensional $Z \subset Y \subset X$, complemented in X.

Examples of subprojective spaces:

- c_0, ℓ_p for $1 \leqslant p < \infty$.
- L_p, $2 \leqslant p < \infty$.

Examples of non-subprojective spaces:

- ℓ_∞.
- L_p, $1 \leqslant p < 2$.

Subprojective spaces

[R.Whitley 1964] A Banach space X is subprojective if, for any infinite dimensional $Y \subset X$, there exists an infinite dimensional $Z \subset Y \subset X$, complemented in X.

Examples of subprojective spaces:

- c_0, ℓ_p for $1 \leqslant p < \infty$.
- L_p, $2 \leqslant p < \infty$.

Examples of non-subprojective spaces:

- ℓ_∞.
- L_p, $1 \leqslant p < 2$.

Subprojective spaces

[R.Whitley 1964] A Banach space X is subprojective if, for any infinite dimensional $Y \subset X$, there exists an infinite dimensional $Z \subset Y \subset X$, complemented in X.

Examples of subprojective spaces:

- c_0, ℓ_p for $1 \leqslant p < \infty$.
- L_p, $2 \leqslant p < \infty$.

Examples of non-subprojective spaces:

- ℓ_∞.
- L_p, $1 \leqslant p < 2$.

Stability of subprojectivity

Proposition

(a) Suppose X and Y are Banach spaces. Then the following are equivalent:

1. *Both X and Y are subprojective.*
2. *$X \oplus Y$ is subprojective.*

(b) Suppose $X_1, X_2, \ldots$ are Banach spaces, and $\mathcal{E}$ is a space with an unconditional basis. Then the following are equivalent:

1. *The spaces $\mathcal{E}, X_1, X_2, \ldots$ are subprojective.*
2. *$(\sum_n X_n)_{\mathcal{E}}$ is subprojective.*

Stability of subprojectivity

Proposition

(a) Suppose X and Y are Banach spaces. Then the following are equivalent:

1. *Both X and Y are subprojective.*
2. *$X \oplus Y$ is subprojective.*

(b) Suppose $X_1, X_2, \ldots$ are Banach spaces, and $\mathcal{E}$ is a space with an unconditional basis. Then the following are equivalent:

1. *The spaces $\mathcal{E}, X_1, X_2, \ldots$ are subprojective.*
2. *$(\sum_n X_n)_{\mathcal{E}}$ is subprojective.*

Subprojectity of spaces of operators

Theorem

Let X be an infinite dimensional Banach space. Then $B(X)$ is not subprojective.

Theorem

Suppose $\mathcal{E}$ is a symmetric sequence space, not containing c_0. Then $\mathfrak{C}_{\mathcal{E}}$ is subprojective if and only if $\mathcal{E}$ is subprojective.

Proposition

Let X and Y be Banach spaces with unconditional bases. If $X^\star$ and Y are subprojective, X is of cotype 2 and Y is of type 2, then $\mathcal{K}(X, Y)$ is subprojective.

Subprojectity of spaces of operators

Theorem

Let X be an infinite dimensional Banach space. Then $B(X)$ is not subprojective.

Theorem

Suppose $\mathcal{E}$ is a symmetric sequence space, not containing c_0. Then $\mathfrak{C}_{\mathcal{E}}$ is subprojective if and only if $\mathcal{E}$ is subprojective.

Proposition

Let X and Y be Banach spaces with unconditional bases. If X^* and Y are subprojective, X is of cotype 2 and Y is of type 2, then $\mathcal{K}(X, Y)$ is subprojective.

Subprojectity of spaces of operators

Theorem

Let X be an infinite dimensional Banach space. Then $B(X)$ is not subprojective.

Theorem

Suppose $\mathcal{E}$ is a symmetric sequence space, not containing c_0. Then $\mathfrak{C}_{\mathcal{E}}$ is subprojective if and only if $\mathcal{E}$ is subprojective.

Proposition

Let X and Y be Banach spaces with unconditional bases. If $X^\star$ and Y are subprojective, X is of cotype 2 and Y is of type 2, then $\mathcal{K}(X, Y)$ is subprojective.

Subprojectity of tensor products

Proposition

For $1 \leqslant p, q < \infty$, $\ell_p \check{\otimes} \ell_q$ and $\ell_p \hat{\otimes} \ell_q$ are subprojective. The same holds if either ℓ_p or ℓ_q is replaced by c_0.

Corollary

The following spaces are subprojective:

- $\mathcal{K}(\ell_p, \ell_q)$, *for* $1 < p < \infty$ *and* $1 \leqslant q < \infty$.
- $\mathcal{K}(c_0, \ell_q)$, *for* $1 \leqslant q < \infty$.
- $\mathcal{K}(\ell_p, c_0)$, *for* $1 < p < \infty$.

Subprojectity of tensor products

Proposition

For $1 \leqslant p, q < \infty$, $\ell_p \check{\otimes} \ell_q$ *and* $\ell_p \hat{\otimes} \ell_q$ *are subprojective. The same holds if either* ℓ_p *or* ℓ_q *is replaced by* c_0.

Corollary

The following spaces are subprojective:

- $\mathcal{K}(\ell_p, \ell_q)$, *for* $1 < p < \infty$ *and* $1 \leqslant q < \infty$.
- $\mathcal{K}(c_0, \ell_q)$, *for* $1 \leqslant q < \infty$.
- $\mathcal{K}(\ell_p, c_0)$, *for* $1 < p < \infty$.

Thank you for your attention!

Many thanks to the organizers of this wonderful conference!