Reflexivity of Banach $C(K)$-Modules via the Reflexivity of Banach Lattices

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X Banach Sp.;

X reflexive:$\Longleftrightarrow X = X'' \Longleftrightarrow$ Ball(X) weakly-compact

X Banach Lat.(**BL**);

X reflexive $\Longleftrightarrow$ (Lozanovskii) no subspace of X is isomorphic to l^1 or c_o.

$\Longleftrightarrow$no sublattice of X is lattice isomorphic to l^1 or c_o (Lotz)

K Compact Hausdorff sp., $C(K)$ complex (or real)-valued continuous functions with sup-norm

$m : C(K) \to \mathcal{L}(X)$ contractive unital algebra homomorphism

X Banach $C(K)$-module ($ax = m(a)(x)$)

$X(x) = cl(C(K)x)$ **cyclic subspace** $\Rightarrow X(x)$ **BL** , with x quasi-interior point, $cl(C(K)_+ x)$ the positive cone

$c_o \nsubseteq X \implies K$ hyperstonian (K Stonian and $C(K)$ dual B Sp), m is (w*,weak-op.)-cont.

$\mathcal{B} = \{e \in C(K) : e = e^2\} \implies m(\mathcal{B})$ **Bade complete Boolean Algebra of Projections** on X

$\implies \mathcal{B}$ complete ($F \subset \mathcal{B} \implies \exists \sup_{e \in F} e \in \mathcal{B}$) and $e_\alpha \uparrow e$ in $\mathcal{B} \implies \|(e - e_\alpha)x\| \to 0$ for all $x \in X$

$\implies$ a cyclic subspace $X(x)$ as BL has order continuos norm and $m(B)$ band projections of BL(Veksler, Schaefer)

X finitely generated:$\Longrightarrow$ $X = X(\{x_1, x_2, \cdots, x_n\})$
(the submodule generated by a finite subset $\{x_1, x_2, \cdots, x_n\}$)

Theorem 1 *Let X be a finitely generated Banach $C(K)$-module. Then X is reflexive $\Longleftrightarrow$ (*) no subspace of X is isomorphic to l^1 or c_o.*

Proof. ($\Leftarrow$) Induction on the number of generators of X.

For $n = 1 \Longrightarrow X = X(x_o)$ BL & (*)$\Longrightarrow$(Lozanovskii) X reflexive.

(**) Assume $n = r \geq 1$ & (*)$\Rightarrow X$ reflexive. Take

$X = X(\{x_o, x_1, x_2, \cdots, x_r\})$ and $Y = X(\{x_1, x_2, \cdots, x_r\})$

(**)$\Longrightarrow Y$ reflexive

$X/Y = X/Y([x_o])$ with $[x_o] = x_o + Y \Longrightarrow X/Y$ BL with order-cts norm, and with $m(\mathcal{B})$ the band projections

$l^1 \not\subseteq X \Longrightarrow l^1 \subsetneq X/Y$. So

X/Y not reflexive$\Longrightarrow$ (Lotz's Theorem) as a sublattice $c_o \subset X/Y$

We show this leads to a contradiction to $c_o \not\subseteq X$.

(This part of the proof is somewhat non-trivial.) ∎

The famous example of the James Space indicates that in general it is not possible to drop the condition that X is finitely generated in Theorem 1.

Nevertheless we will consider next extending Theorem 1 to Banach $C(X)$-modules that are infinitely generated while staying close to being finitely generated.

In this case we specialize to when $\mathcal{B}$ is a Bade complete Boolean algebra of projections on X immediately.

$\mathcal{B}$ of **uniform multiplicity** n **on** X $:\Longleftrightarrow$ $\exists$ disjoint $\{e_\alpha\} \subset \mathcal{B}$ such that , for each α, $ee_\alpha = e \in \mathcal{B} \Rightarrow eX$ is generated by a minimum of n elements and $\mathbf{sup}\, e_\alpha = \mathbf{1}$.

(Rall) B is of uniform multiplicity one on $X \Longrightarrow X$ BL with order-cts norm and $\mathcal{B}$ the band projections of the BL

Corollary 2 *Let $\mathcal{B}$ be of uniform multiplicity n on X. Then X is reflexive $\Longleftrightarrow$ (*) no subspace of X is isomorphic to l^1 or c_o.*

Proof. ($\Leftarrow$) By induction on n.

$n = 1$ by Rall's Theorem X is BL. Then (*) and Lozanovskii's Theorem $\Longrightarrow X$ reflexive.

The rest of the proof mimics the proof of Theorem 1. ∎

$\mathcal{B}$ is of **finite multiplicity on** X $:\Longleftrightarrow \exists$ disjoint $\{e_\alpha\} \subset \mathcal{B}$ such that , for each α, $e_\alpha X$ is generated by a minimum of n_α elements and $\sup e_\alpha = 1$. ($\{n_\alpha\}$ may be unbounded)

(Bade) $\mathcal{B}$ is of finite multiplicity on $X \implies \exists$ disjoint $\{e_n\} \subset \mathcal{B}$ such that $\mathcal{B}$ is of uniform multiplicity n on e_nX and $\sup e_n = 1$.

Theorem 3 *Let $\mathcal{B}$ be of finite multiplicity on X. Then X is reflexive $\iff$ (*) no subspace of X is isomorphic to l^1 or c_o.*

Proof. $\chi_n := e_1 + e_2 + \cdots + e_n \in \mathcal{B}$ for each n

Corollary 2 $\implies e_nX$ and χ_nX reflexive.

Assume $e_n \neq 0$ for all n (An idea of James motivates the main step)

(***) $l^1 \nsubseteq X \Longrightarrow \|(1 - \chi_n)f\| \rightarrow 0$ for each $f \in X'$

$l^1 \nsubseteq X \Longrightarrow c_o \nsubseteq X'$ (Bessaga and Pelczynski) and

$\mathcal{B}$ Bade complete on $X \Longrightarrow cl(span(\mathcal{B}^*))$ weak*-operator closed in $\mathcal{L}(X')$ (Orhon)

together imply $\mathcal{B}^*$ Bade complete on $X' \Longrightarrow$ (***)

Then we show Ball(X) is weakly-compact ■