Matrix monotone functions and a generalized Powers-Størmer inequality

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Introduction

Let φ be a normal state on the algebra $B(H)$ of all bounded operators on a Hilbert space H, f a strictly positive, continuous function on $(0,\infty)$, and let g be a function on $(0,\infty)$ defined by $g(t) = \frac{t}{f(t)}$. We will give characterizations of matrix and operator monotonicity by the following generalized Powers-Størmer inequality:

$$\varphi(A+B) - \varphi(|A-B|) \leq 2\varphi(f(A)^{\frac{1}{2}} g(B) f(A)^{\frac{1}{2}}),$$

whenever A, B are positive invertible operators in $B(H)$.

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When $\dim H < \infty$ and φ is the canonical trace, and $f(t) = t^{\frac{1}{2}}$, Powers and Størmer proved the inequality in 1970.
When $f(t) = t^s \quad (0 \leq s \leq 1)$, K. M. R.Audenaert, J. Calsamiglia, LI. Masanes, R. Munoz-Tapia, A. Acin, E. Bagan, F. Verstraete proved the inequality in 2006.

Quantam information theory

One of the most basic tasks in quantum statistics is the discrimination of two different quantum states. In the quantum hypothesis testing problem, one has to decide between two states of a system. The state $\rho_0$is the null hypothesis and ρ_1 is alternative hypothesis.

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The problem is to decide which hypothesis is true. The decision is performed by a two-valued measurement $\{T, I - T\}$, where $0 \leq T \leq I$ is an observable. T corresponds to the acceptance of ρ_0 and $I - T$ corresponds to the acceptance of ρ_1. T is called a test.

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Total error probability:

$$\rho_0, \rho_1 : \textit{ hypothetic states on } \mathbf{C}^d$$

$$: \text{density matrix on } \mathbf{C}^d, \text{that is}$$

$$\rho_i \geq 0, \mathrm{Tr}(\rho_i) = 1 \ (i = 0, 1)$$

$$T = \{T_0, T_1\} : \textit{ quantum multiple test}$$

$$: \ d \times d \text{ positive matrices } T_0 + T_1 = I$$

$$\mathrm{Succ}_i(T) := \mathrm{Tr}(\rho_i T_i) \ (i = 0, 1)$$

$$: \text{the probability of acceptance of } \rho_i$$

$$\mathrm{Err}_i(T) := 1 - \mathrm{Succ}_i(T) = \mathrm{Tr}(\rho_i(1 - T_i))$$

$$: \text{the probability that hypothesis } i \text{ is true}$$

$$\text{but the hypothesis } i + 1 \text{ is acceptance}$$

$$\mathrm{Err}(T) := \frac{1}{2}\mathrm{Tr}(\rho_0 T_1) + \frac{1}{2}\mathrm{Tr}(\rho_1 T_0)$$

$$= \frac{1}{2}\{1 - \mathrm{Tr}(T_0(\rho_0 - \rho_1))\} \ \ : \text{Total error probability}$$

Assymptotic error exponent for ρ_0 and ρ_1

$$\forall n \in \mathbf{N} \quad T_{(n)} : d^n \times d^n \text{quantum multiple test}$$
$$\mathrm{Err}_n(T_{(n)}) := \frac{1}{2}\{1 - \mathrm{Tr}(T_{(n)}(\rho_0^{\otimes n} - \rho_1^{\otimes n}))\}$$

If the limit $\lim_{n\to\infty} -\frac{1}{n}\log \mathrm{Err}_n(T_{(n)})$ exists, we refer to it as *the asymptotic error exponent*.

Theorem (M. Nussbaum and A. Szkola 2006, K. M. R. Audenaert, et al. 2006)

Let $\{\rho_0, \rho_1\}$ be hypothetic states on $\mathbf{C}^d$, $T_{(n)}$ be quantum multiple test, and $Q_{(n)}$ be a support projections on $(\rho_0^{\otimes n} - \rho_1^{\otimes n})$. Then one has

(i) (M. Nussbaum and A. Szkola)

$$\liminf_{n\to\infty} \frac{1}{n} \log \mathrm{Err}_n(T_{(n)}) \geq \inf\{\log \mathrm{Tr}(\rho_0^{1-s}\rho_1^s) \mid 0 \leq s \leq 1\}$$

(ii) (K. M. R. Audenaert, et al.)

$$\limsup_{n\to\infty} \frac{1}{n} \log \mathrm{Err}_n(Q_{(n)}) \leq \inf\{\log \mathrm{Tr}(\rho_0^{1-s}\rho_1^s) \mid 0 \leq s \leq 1\}$$

In the proof of Theorem 0.1(ii) the following inequality played a key role.

Theorem (K. M. R. Audenaert et al. 2011)

For any positive matrices A and B on $\mathbf{C}^d$ we have

$$\frac{1}{2}(\operatorname{Tr} A + \operatorname{Tr} B - \operatorname{Tr}|A - B|) \leq \operatorname{Tr}(A^{1-s}B^s) \ (s \in [0, 1]).$$

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If we consider a function $f(t) = t^{1-s}$ and $g(t) = t^s = \dfrac{t}{f(t)}$, then both functions f and g are operator monotone.

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If we consider a function $f(t) = t^{1-s}$ and $g(t) = t^s = \frac{t}{f(t)}$, then both functions f and g are operator monotone.
The inequality, then, can be reformed by
$\operatorname{Tr} A + \operatorname{Tr} B - \operatorname{Tr} |A - B| \leq 2 \operatorname{Tr}(f(A)^{\frac{1}{2}} g(B) f(A)^{\frac{1}{2}})$.

Main result

Theorem

Let φ be a normal state on $B(H)$, f be a strictly positive, continuous function on $(0,\infty)$, and let g be a function on $(0,\infty)$ defined by $g(t) = \frac{t}{f(t)}$. Consider the following inequality : For any positive invertible $A, B \in B(H)$

$$\varphi(A+B) - \varphi(|A-B|) \leq 2\varphi(f(A)^{\frac{1}{2}} g(B) f(A)^{\frac{1}{2}}).$$

1. When $\dim H = n < \infty$, if φ is the canonical trace Tr and f is $(n+1)$-concave, then the inequality holds.
2. If the inequality holds true for any positive invertible A, B, then the functions f and g are operator monotone.

Double piling structure of matrix functions

Definition

Let $I \subset \mathbf{R}$ be an interval. A function $f : I \to \mathbf{R}$ is said to be *matrix convex of order n* or *n-convex* in short (resp. *matrix concave of order n* or *n-concave*) whenever the inequality

$$f(\lambda A + (1-\lambda)B) \leq \lambda f(A) + (1-\lambda)f(B), \ \ \lambda \in [0,1]$$

(resp. $f(\lambda A + (1-\lambda)B) \geq \lambda f(A) + (1-\lambda)f(B), \ \ \lambda \in [0,1]$) holds for every pair of selfadjoint matrices $A, B \in M_n$ such that all eigenvalues of A and B are contained in I.

Definition

A function $f : I \to \mathbf{R}$ is said to be *matrix monotone functions* are similarly defined as the inequality

$$A \leq B \Longrightarrow f(A) \leq f(B)$$

for any pair of selfadjoint matrices $A, B \in M_n$ such that $A \leq B$ and all eigenvalues of A and B are contained in I.

We call a function f *operator convex* (resp. *operator concave*) if for each $k \in \mathbb{N}$, f is k-convex (resp. k-concave) and *operator monotone* if for each $k \in \mathbb{N}$ f is k-monotone.

1. $P_n(I)$: the spaces of n-monotone functions
2. $P_\infty(I)$: the space of operator monotone functions
3. $K_n(I)$: the space of n-convex functions
4. $K_\infty(I)$: the space of operator convex functions

Then we have

$$P_1(I) \supseteq \cdots \supseteq P_{n-1}(I) \supseteq P_n(I) \supseteq P_{n+1}(I) \supseteq \cdots \supseteq P_\infty(I)$$
$$K_1(I) \supseteq \cdots \supseteq K_{n-1}(I) \supseteq K_n(I) \supseteq K_{n+1}(I) \supseteq \cdots \supseteq K_\infty(I)$$

$P_{n+1}(I) \subsetneq P_n(I)$ $K_{n+1}(I) \subsetneq K_n(I)$
$P_\infty = \cap_{n=1}^\infty P_n(I)$ $K_\infty = \cap_{n=1}^\infty K_n(I)$

Theorem

Let consider the following three assertions.

(i) $f(0) \leq 0$ and f is n-convex in $[0, \alpha)$,

(ii) For each matrix a with its spectrum in $[0, \alpha)$ and a contraction c in the matrix algebra M_n,

$$f(c^\star ac) \leq c^\star f(a)c,$$

(iii) The function $\frac{f(t)}{t}$ is n-monotone in $(0, \alpha)$.

1. (Hansen-Pedersen:1985) Three assertions are equivalent if f is operator convex. In this case a function $\frac{f(t)}{t}$ is operator monotone.
2. (O-Tomiyama:2009)

$$\text{(i)}_{n+1} \prec \text{(ii)}_n \sim \text{(iii)}_n \prec \text{(i)}_{[\frac{n}{2}]}.$$

Here we denotion $(A)_m \prec (B)_n$ means that "if (A) holds for the matrix algebra M_m, then (B) holds for the matrix algebra M_n".

The following result is essentially proved in [Hansen-Pedersen:1982].

Proposition

Let f be a strictly positive, continuous function on $[0, \infty)$. If f is $2n$-monotone, the function $g(t) = \frac{t}{f(t)}$ is n-monotone.

Using an idea in [O-Tomiyama: 2009] we can show the following result.

Proposition (D. T. Hoa-O-J. Tomiyama 2012)

Let $n \in \mathbf{N}$ and $f : [0, \alpha) \to \mathbf{R}$ be a continuous function for some $\alpha > 0$ such that $0 \notin f([0, \alpha))$. Let us consider the following assertions:

(iv) f is n-concave with $f(0) \geq 0$.

(v) For each matrix a with its spectrum in $[0, \alpha)$ and a contraction c in the matrix algebra M_n,
$$f(c^\star ac) \geq c^\star f(a)c,$$

(vi) The function $\frac{t}{f(t)}$ $(= g(t))$ is n-monotone in $(0, \alpha)$.

We have, then, $(\mathrm{iv})_{n+1} \prec (\mathrm{v})_n \sim (\mathrm{vi})_n \prec (\mathrm{i})_{[\frac{n}{2}]}$.

Generalized Powers-Størmer inequality

Theorem (D. T. Hoa-O-H. M. Toan, D. T. Hoa-O-J. Tomiyama 2012)

Let f be a $2n$-monotone function (or $(n+1)$-concave function) on $[0,\infty)$ such that $f((0,\infty)) \subset (0,\infty)$. Then for any pair of positive matrices $A, B \in M_n(\mathbf{C})$

$$\mathrm{Tr}(A) + \mathrm{Tr}(B) - \mathrm{Tr}(|A-B|) \leq 2\,\mathrm{Tr}(f(A)^{\frac{1}{2}} g(B) f(A)^{\frac{1}{2}}),$$

where $g(t) = \frac{t}{f(t)}$.

Sketch of the proof:

A, B : positive matrices

$A - B = (A-B)_+ - (A-B)_- = P - Q,$

$|A-B| = P + Q.$

We may show that

$\mathrm{Tr}(A) - \mathrm{Tr}(f(A)^{\frac{1}{2}} g(B) f(A)^{\frac{1}{2}}) \le \mathrm{Tr}(P)$ holds as follows:

$$\begin{aligned}
&\mathrm{Tr}(A) - \mathrm{Tr}(f(A)^{\frac{1}{2}} g(B) f(A)^{\frac{1}{2}}) \\
&= \mathrm{Tr}(f(A)^{\frac{1}{2}} g(A) f(A)^{\frac{1}{2}}) - \mathrm{Tr}(f(A)^{\frac{1}{2}} g(B) f(A)^{\frac{1}{2}}) \\
&\le \mathrm{Tr}(f(A)^{\frac{1}{2}} g(B+P) f(A)^{\frac{1}{2}}) - \mathrm{Tr}(f(A)^{\frac{1}{2}} g(B) f(A)^{\frac{1}{2}}) \\
&\le \mathrm{Tr}(f(B+P)^{\frac{1}{2}} (g(B+P) - g(B)) f(B+P)^{\frac{1}{2}}) \\
&\le \mathrm{Tr}(f(B+P)^{\frac{1}{2}} g(B+P) f(B+P)^{\frac{1}{2}}) - \mathrm{Tr}(f(B)^{\frac{1}{2}} g(B) f(B)^{\frac{1}{2}}) \\
&= \mathrm{Tr}(P)
\end{aligned}$$

Corollary

Let f be an operator monotone function on $[0, \infty)$ such that $f((0, \infty)) \subset (0, \infty)$. Then for any pair of positive matrices $A, B \in M_n(\mathbf{C})$

$$\operatorname{Tr}(A) + \operatorname{Tr}(B) - \operatorname{Tr}(|A - B|) \leq 2 \operatorname{Tr}(f(A)^{\frac{1}{2}} g(B) f(A)^{\frac{1}{2}}),$$

where $g(t) = \frac{t}{f(t)}$.

Example

Let $f(t) = t^2$ on $(0, \infty)$. It is well-known that f is not 2-monotone. We now show that the function f does not satisfy the inequality in Theorem. Indeed, let us consider the following matrices

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}.$$

Then we have

$$AB^{-1}A = \frac{2}{3}A.$$

Set $\tilde{A} = A \oplus \mathrm{diag}(\underbrace{1, \cdots, 1}_{n-2}), \tilde{B} = B \oplus \mathrm{diag}(\underbrace{1, \cdots, 1}_{n-2})$ in M_n.

Example

Then, $\tilde{A} \le \tilde{B}$ and for any positive linear function φ on M_n

$$\begin{aligned}\varphi(f(\tilde{A})^{\frac{1}{2}}g(\tilde{B})f(\tilde{A})^{\frac{1}{2}}) &= \varphi(\tilde{A}\tilde{B}^{-1}\tilde{A}) \\ &= \varphi(\frac{2}{3}A \oplus \mathrm{diag}(\underbrace{1, \cdots, 1}_{n-2})) \\ &< \varphi(A \oplus \mathrm{diag}(\underbrace{1, \cdots, 1}_{n-2})) \\ &= \varphi(\tilde{A}).\end{aligned}$$

On the contrary, since $\tilde{A} \le \tilde{B}$, from the inequality we have $\varphi(\tilde{A}) \le \varphi(f(\tilde{A})^{\frac{1}{2}}g(\tilde{B})f(\tilde{A})^{\frac{1}{2}})$, and we have a contradiction. □

Applications

For matrices $A, B \in M_n^+$ let us denote

$$Q(A,B) = \min_{s\in[0,1]} \mathrm{Tr}(A^{(1-s)/2}B^sA^{(1-s)/2}) \qquad (1)$$

and

$$Q_{\mathcal{F}_{2n}}(A,B) = \inf_{f\in\mathcal{F}_{2n}} \mathrm{Tr}(f(A)^{\frac{1}{2}}g(B)f(A)^{\frac{1}{2}}), \qquad (2)$$

where $\mathcal{F}_{2n}$ is the set of all $2n$-monotone functions on $[0,+\infty)$ satisfy condition of the Theorem and
$g(t) = tf(t)^{-1}$ $(t \in [0,+\infty))$.
Note that the function $f(t) = t^s$ $(t \in [0,+\infty))$ satisfies the conditions of Theorem. Since the class of $2n$-monotone functions is large enough, we know that $Q_{\mathcal{F}_{2n}}(A,B) \leq Q(A,B)$.

Hence, we hope on finding another $2n$-monotone function f on $[0, +\infty)$ such that

$$\mathrm{Tr}(f(A)^{\frac{1}{2}} g(B) f(A)^{\frac{1}{2}}) < Q(A, B). \tag{3}$$

If we can find such a function, then we may get smaller upper bound than what is used in quantum hypothesis testing. For example, considering the trace distance $T(A, B) = \dfrac{\mathrm{Tr}(|A - B|)}{2}$, we might have the following better estimate

$$\frac{1}{2}\mathrm{Tr}(A+B) - Q_{\mathcal{F}_{2n}}(A, B) \leq T(A, B) \leq \sqrt{\{\frac{1}{2}\mathrm{Tr}(A+B)\}^2 - Q_{\mathcal{F}_{2n}}(A, B)^2}.$$

Theorem

Let τ be a tracial functional on a C^*-algebra $\mathcal{A}$, f be a strictly positive, operator monotone function on $[0,\infty)$. Then for any pair of positive elements $A, B \in \mathcal{A}$

$$\tau(A) + \tau(B) - \tau(|A-B|) \leq 2\tau(f(A)^{\frac{1}{2}} g(B) f(A)^{\frac{1}{2}}), \qquad (4)$$

where $g(t) = tf(t)^{-1}$.

Characterizations of the trace property

Lemma

Let φ be a positive linear functional on M_n and f be a continuous function on $[0, \infty)$ such that $f(0) = 0$ and $f((0, \infty)) \subset (0, \infty)$. If the following inequality

$$\varphi(A + B) - \varphi(|A - B|) \leq 2\varphi(f(A)^{\frac{1}{2}} g(B) f(A)^{\frac{1}{2}}) \qquad (5)$$

holds true for all $A, B \in M_n^+$, then φ should be a positive scalar multiple of the canonical trace Tr on M_n, where

$$g(t) = \begin{cases} \frac{t}{f(t)} & (t \in (0, \infty)) \\ 0 & (t = 0) \end{cases} .$$

Theorem

Let φ be a positive linear functional on a C^*-algebra $\mathcal{A}$ and f be a continuous function on $[0, \infty)$ such that $f(0) = 0$ and $f((0, \infty)) \subset (0, \infty)$. If the following inequality

$$\varphi(A) + \varphi(B) - \varphi(|A - B|) \leq 2\varphi(f(A)^{\frac{1}{2}} g(B) f(A)^{\frac{1}{2}}) \qquad (6)$$

holds true for any pair $A, B \in \mathcal{A}^+$, then φ is a tracial functional, where $g(t) = \begin{cases} \frac{t}{f(t)} & (t \in (0, \infty)) \\ 0 & (t = 0) \end{cases}$.

Proposition

Let $n \in \mathbf{N}$ and φ be a positive linear functional on $M_n(\mathbf{C})$. Let f be a strictly positive, continuous function on $(0, \infty)$, and let g be a function on $(0, \infty)$ defined by $g(t) = \frac{t}{f(t)}$, g is differentiable and strictly increasing on $(0, \infty)$. Suppose that for any positive invertible $A, B \in M_n(C)$ $(0 < A \leq B)$

$$\varphi(A) \leq \varphi(f(A)^{\frac{1}{2}} g(B) f(A)^{\frac{1}{2}}). \tag{7}$$

Then φ has the trace property if g satisfies the condition

$$\inf_{\lambda > \mu} \frac{\sqrt{g'(\lambda) g'(\mu)}}{\frac{g(\lambda) - g(\mu)}{\lambda - \mu}} = 0. \tag{8}$$

Example

For $g(x) = t^2$ (i.e. $f(t) = 1/t$) on $(0, \infty)$ which satisfies the condition (8), and for any $n \in \mathbb{N}$

$$\mathrm{Tr}(A) \leq \mathrm{Tr}(f(A)^{\frac{1}{2}} g(B) f(A)^{\frac{1}{2}})$$

whenever $0 < A \leq B$ in M_n.

Example

For $g(x) = e^x$ on $(0, \infty)$ which satisfies the condition (8), and for any $n \in \mathbb{N}$, we have

$$\mathrm{Tr}(A) \leq \mathrm{Tr}((Ae^{-A})^{\frac{1}{2}} e^B (Ae^{-A})^{\frac{1}{2}})$$

whenever $0 < A \leq B$ in M_n.

Characterization of operator monotonicity

Theorem

Let φ be a normal state on $B(H)$, f be a strictly positive, continuous function on $(0, \infty)$, and let g be a function on $(0, \infty)$ defined by $g(t) = \frac{t}{f(t)}$. Suppose that for any positive invertible $A, B \in B(H)$

$$\varphi(A + B) - \varphi(|A - B|) \leq 2\varphi(f(A)^{\frac{1}{2}} g(B) f(A)^{\frac{1}{2}}).$$

Then both the functions f and g on $(0, \infty)$ are operator monotone.

ありがとうございました

Thank You