A vector lattiice version of Rådström's embedding theorem

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Outline

1. Motivation
   - The basic problem that we studied
   - Rådström's embedding theorem

2. Our Results/Contribution
   - Main Results
   - Applications

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Our motivation.

- Let X be a Banach space and $\mathcal{P}_0(X)$ be the collection of nonempty subsets of X.
- $A + B = \{a + b : a \in A, b \in B\}$ and $\lambda A = \{\lambda a : a \in A\}$, for $A, B \subseteq X$, λ a scalar.
- Given $A \in \mathcal{P}_0(X)$, it is not always possible to find an additive inverse of A.

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The Hausdorff distance

Let (X, d) be a metric space. If $A \in \mathcal{P}(X)$ and $x \in X$, the distance between x and A is defined by

$$d(x, A) = \inf\{d(x, y) : y \in A\}.$$

If $A, B \in \mathcal{P}(X)$, then the Hausdorff distance d_H between A and B is defined by

$$d_H(A, B) = \sup_{a \in A} d(a, B) \vee \sup_{b \in B} d(b, A).$$

In the special case where $B = \{0\}$, let $\|A\|_H = d_H(A, \{0\})$.

Rådström's embedding theorem.

What follows from Rådström's theorem, in [5], that

- a *'near vector space'*, can be embedded into a vector space.
- if the 'near vector space' is endowed with a metric compatible with the addition and multiplication by positive scalars defined on the near vector space, then the embedding space can be normed and the embedding also preserves distance.

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Implications of the embedding theorem

- Rådström described the conditions for under which a 'near vector space' X can be embedded into a vector space.
- What is required is a a cancellation law:

$$A + B = A + C \Rightarrow B = C,$$

for $A, B, C \in X$

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Examples.

- $\mathrm{cbf}(X)$ of nonempty convex bounded closed subsets of X,
- $\mathrm{cwk}(X)$ of nonempty convex weakly compact subsets of X,
- $\mathrm{ck}(X)$ of nonempty convex compact subsets of X,

for X a Banach space with suitable addition defined.

For the latter two examples, ordinary set addition as defined previously is suitable but in the case of $\mathrm{cbf}(X)$ we do not have closure under addition.

We can overcome this by defining addition $\oplus$ as follows:

$$A \oplus B = \overline{A + B},$$

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The definition of a near vector space

Let S be a nonempty set.

1. [1] Then S is said to be a *near vector space*, provided that: addition $+ : S \times S \to S$ is defined such that $(S, +)$ is a cancellative commutative semigroup; i.e., for all $x, y, z \in S$:

   (a) $x + z = y + z \Rightarrow x = y$,
   (b) $x + y = y + x$,
   (c) $(x + y) + z = x + (y + z)$,

   and multiplication $\cdot : \mathbb{R}_+ \times S \to S$ by positive scalars is defined such that for all $x, y \in S$ and $\lambda, \delta \in \mathbb{R}_+$:

   (d) $\lambda x + \lambda y = \lambda(x + y)$,
   (e) $(\lambda + \delta)x = \lambda x + \delta x$,
   (f) $(\lambda\delta)x = \lambda(\delta x)$,
   (g) $1x = x$.

1. [2] If S is a near vector space and $d\colon S \times S \to \mathbb{R}_+$ is a metric on S, then d is said to be an *invariant metric* on S, provided that:
   (h) the two mappings
   $$+ : S \times S \to S, \quad (x, y) \mapsto x + y, \text{ and}$$
   $$\cdot : \mathbb{R}_+ \times S \to S, \quad (t, y) \mapsto t \cdot x,$$
   are continuous when $\mathbb{R}_+$ is equipped with the subspace topology (as a subspace of b with the usual topology), S is equipped with the metric topology induced by d, and $\mathbb{R}_+ \times S$ and $S \times S$ are equipped with the corresponding product topologies.
   (i) $d(\lambda x, \lambda y) = \lambda d(x, y)$, for all $\lambda \in \mathbb{R}_+$ and $x, y \in S$,
   (j) $d(x + z, y + z) = d(x, y)$, for all $x, y, z \in S$.

The embedding theorem.

The following is a special case of what Rådström proved in [5, Theorem 1].

Theorem

Let S be a near vector space.

1. *There exist a vector space $R(S)$ and a map $j\colon S \to R(S)$ such that*
   1. *j is injective,*
   2. $j(\alpha x + \beta y) = \alpha j(x) + \beta j(y)\ \forall \alpha, \beta \in \mathbb{R}_+$ *and* $x, y \in S$,
   3. $R(S) = j(S) - j(S) := \{j(x) - j(y) : x, y \in S\}$.
2. *If $d\colon S \times S \to \mathbb{R}$ is an invariant metric, then there exists a norm $\|\cdot\|_d$ on $R(S)$ such that*

$$d(x, y) = \|j(x) - j(y)\|_d, \text{ for all } x, y \in S.$$

Rådström's embedding theorem proof

Consider $S \times S$ and define $\sim$ on $S \times S$ by

$$(x, y) \sim (x_1, y_1) \Leftrightarrow x + y_1 = x_1 + y.$$

Then $\sim$ is an equivalence relation on $S \times S$. Let

$$[x, y] := \{(x_1, y_1) \in S \times S : (x, y) \sim (x_1, y_1)\}.$$

On the quotient $R(S) := (S \times S)/_\sim, = \{[x, y] : (x, y) \in S \times S\}$, define addition by

$$[x, y] + [x_1, y_1] = [x + x_1, y + y_1].$$

Then $R(S)$ is an abelian group with additive identity $[x, x]$ and additive inverse

$$-[x, y] := [y, x], \text{ for any } (x, y) \in S \times S.$$

If multiplication by positive scalars is defined on S with the properties as stated above, define scalar multiplication $\cdot : \mathbb{R} \times R(S) \to R(S)$ by

$$\lambda \cdot [x, y] := \begin{cases} [\lambda x, \lambda y] & \lambda \in \mathbb{R}_+ \\ [-\lambda y, -\lambda x] & -\lambda \in \mathbb{R}_+ . \end{cases}$$

Then $R(S)$ is a vector space.
If d is an invariant metric on S, then $\| \cdot \|_d$, defined by

$$\| [x, y] \|_d := d(x, y), \text{ for all } [x, y] \in R(S),$$

is a norm on $R(S)$ with the desired property.
The map $j\colon S \to R(S)$, defined by

$$j(x) = [x + z, z], \text{ for all } x \in S,$$

for any $z \in S$, has the desired properties.

Our problem

We aimed to answer the following questions.

- What results can we obtain if the near vector space was equipped with a partial ordering?
- What happens if the near vector space is equipped with a 'order unit'?
- When can you obtain a $C(\Omega)$ embedding?
- Finally, which conditions on the near vector space are required for an $L^p(\mu)$ embedding?

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Near vector lattices

Definition

Let S be a near vector space.

(a) If $(S, \leq)$ is a partially ordered set such that $\leq$ is compatible with addition and multiplication by positive scalars; i.e.,

  (i) $x \leq y \;\Rightarrow\; x + z \leq y + z$, and
  (ii) $x \leq y \;\Rightarrow\; \alpha x \leq \alpha y$, for all $\alpha \in \mathbb{R}_+$,

  then S is called an *ordered near vector space*.

(b) If S is an ordered near vector space and $(S, \leq)$ is a join-semilattice for which

$$(x \vee y) + z = (x + z) \vee (y + z), \text{ for all } x, y, z \in S,$$

  then S is called a *near vector lattice*.

Examples

Let X be a Banach space.

1. $(\mathrm{cbf}(X), \oplus)$ satisfies the cancellation law (C). It is then easily verified, that $(\mathrm{cbf}(X), \oplus, \cdot\,, \subset)$ satisfies the other properties required to be an ordered vector space. Furthermore, $(\mathrm{cbf}(X), \subset)$ is a join-semilattice with join given by
$$K_1 \vee K_2 = \overline{\mathrm{co}}(K_1 \cup K_2)$$
(where the latter denotes the norm closure of the convex hull of $K_1 \cup K_2$).
Then, $\mathrm{cbf}(X)$ is a near vector lattice.
2. It is well-known that if $K_1, K_2 \in \mathrm{ck}(X)$, then $K_1 \oplus K_2 = K_1 + K_2 \in \mathrm{ck}(X)$.
If $K_1, K_2 \in \mathrm{ck}(X)$, then $K_1 \vee K_2 \in \mathrm{ck}(X)$. It follows easily that $\mathrm{ck}(X)$ is a near vector lattice with respect to the operations and order induced by $\mathrm{cwk}(X)$.

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Properties of near vector lattices

Let S be a near vector lattice. Then $R(S)$ is a vector lattice, with positive cone $R(S)_+ := \{[x, y] : y \leq x\}$, in which the following formulas hold:

1. $[x, y]^+ = [x \vee y, y]$,
2. $[x, y]^- = [x \vee y, x]$,
3. $|\,[x, y]\,| = [2(x \vee y), x + y]$,
4. $[x, y] \vee [x_1, y_1] = [(x_1 + y) \vee (x + y_1), y + y_1]$,
5. $[x, y] \wedge [x_1, y_1] = [x + x_1, (x_1 + y) \vee (x + y_1)]$.

Zero elements

Definition

Let S be an ordered near vector space [near vector lattice] which also satisfies

(Z) there exists $0 \in S$ such that $x + 0 = x$, for all $x \in S$ and $\lambda 0 = 0$, for all $\lambda \in \mathbb{R}_+$.

Then S is said to be an *ordered near vector space [near vector lattice] with a zero*.

Some important isomorphisms

Theorem

Let E be a vector lattice. Then $J\colon R(E_+) \to E$, defined by

$$J([x,y]) = x - y, \text{ for all } x, y \in E_+,$$

is a vector lattice isomorphism.

Corollary

Let E be a Riesz normed vector lattice. Then

1. $d_{\|\cdot\|}\colon E_+ \times E_+ \to \mathbb{R}_+$, *defined by* $d_{\|\cdot\|}(x,y) = \|x - y\|$, *for all* $x, y \in E_+$, *is a Riesz metric on* E_+, *and*
2. *the map J, as in the embedding theorem, is a surjective vector lattice and isometric isomorphism.*

Riesz (semi)norms

Definition

A Riesz (semi)norm $\|\cdot\|$ on a vector lattice E is called an *M-(semi)norm* on E, provided that

(M) $\|x \vee y\| = \max\{\|x\|, \|y\|\}$, for all $x, y \in E_+$.

In this case we call $E = (E, \|\cdot\|)$ an *M-(semi)normed space*

Definition

A Riesz (semi)norm on a vector lattice E is called an *L-(semi)norm*, provided that

$$\|x + y\| = \|x\| + \|y\|, \text{ for all } x, y \in E_+.$$

A vector lattice equipped with an L-(semi)norm is said to be an L-(semi)normed space.

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Theorem (Kakutani's (*M*)-space representation theorem, [2])

Let E be any M-normed space with an order unit, then there exists a compact Hausdorff space Ω such that E is isometric and lattice isomorphic to the space $C(\Omega)$ of all continuous real-valued functions $f(\omega)$ defined on Ω.

Order unit for a near vector lattice and examples

Definition

Let S be an ordered near vector space. If $e \in S$ has the property that

(ou) for all $x, y \in S$, there exists $K \in \mathbb{R}_+$ such that $x \leq Ke + y$ and $y \leq Ke + x$,

then e is called an *order unit* of S.

Lemma

Let S be an ordered near vector space which has an order unit e. Then

1. *$[e + z, z]$ is an order unit of the ordered vector space $R(S)$, for any fixed $z \in S$, and*
2. *$R(S)$ is Archimedean if S has a zero and is Archimedean.*

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Examples

Let X be a Banach space.

1. Then B_X is an order unit of the near vector lattice $\mathrm{cbf}(X)$.
2. If X is reflexive, then B_X is an order unit of the near vector lattice $\mathrm{cwk}(X)$.
3. If X is finite dimensional, then B_X is an order unit of the near vector lattice $\mathrm{ck}(X)$.

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A $C(\Omega)$ embedding

Theorem

Let S be an Archimedean near vector lattice with a zero and an order unit $e \in S$. Then there exist a compact Hausdorff space Ω and a map $K\colon S \to C(\Omega)$ such that:

1. *K is injective.*
2. *$K(\alpha x + \beta y) = \alpha K(x) + \beta K(y)$, for all $x, y \in S$ and $\alpha, \beta \in \mathbb{R}_+$.*
3. *$K(S) - K(S)$ is norm dense in $C(\Omega)$.*
4. *$h_e(x, y) = \|K(x) - K(y)\|_\infty$, for all $x, y \in S$.*
5. *$K(x \vee y) = K(x) \vee K(y)$, for all $x, y \in S$.*
6. *$K(e) = \mathbf{1}$.*

An $L^p(\mu)$ embedding

Lemma

Let S be a near vector lattice and $d\colon S \times S \to \mathbb{R}_+$ an invariant metric. Then d is a Riesz metric and

(L_d) $d(x + x_1, y + y_1) = d(x, y) + d(x_1, y_1)$, *for all* $x, y, x_1, y_1 \in S$ *such that* $y \leq x$ *and* $y_1 \leq x_1$

if and only if $\|\cdot\|_d$ *is an L-norm on* $R(S)$.

Theorem

Let S be an Archimedean near vector lattice with 0 as zero and e as an order unit. If $d\colon S \times S \to \mathbb{R}_+$ is a Riesz metric with property (L_d) and $d(e,0) = 1$, then there exist a probability space (Ω, Σ, μ) and a map $K\colon S \to L^1(\mu)$ such that:

1. *K is injective.*
2. *$K(\alpha x + \beta y) = \alpha K(x) + \beta K(y)$, for all $x, y \in S$ and $\alpha, \beta \in \mathbb{R}_+$.*
3. *$K(S) - K(S)$ is norm dense in $L^1(\mu)$.*
4. *$d(x,y) = \|K(x) - K(y)\|_1$, for all $x, y \in S$.*
5. *$K(x \vee y) = K(x) \vee K(y)$, for all $x, y \in S$.*
6. *$K(e) = \mathbf{1}$, where the latter denotes a function which is one almost everywhere.*

Outline

A version of Doob's decomposition theorem [see [4], p.159]

Let $X^* = \{x^* : X \to \mathbb{R} : x^*$ is linear and continuous$\}$. For every bounded subset C of X and each $x^* \in X^*$, let

$$s(x^*, C) := \sup\{x^*(x) : x \in C\}.$$

Theorem

Let (Ω, Σ, μ) be a measure space and X a separable real Banach space. Let (F_i, Σ_i) be a set-valued submartingale in $\mathcal{L}^1[\Omega, \mathrm{cf}(X)]$; If there exists $B \in \Sigma$ with $\mu(B) = 0$ such that for any $\omega \notin B$ and all $i \in \mathbb{N}$,

(i) $s(\,\cdot\,, \mathcal{E}[F_i|\Sigma_{i-1}](\omega)) - s(\,\cdot\,, F_{i-1}(\omega))$, *and*

(ii) $s(\,\cdot\,, F_i(\omega)) - s(\,\cdot\,, \mathcal{E}[F_i|\Sigma_{i-1}](\omega))$,

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Theorem

as

$$F_i(\omega) = M_i(\omega) \oplus A_i(\omega), \text{ for all } \omega \notin B,$$

where (M_i, Σ_i) *is a set-valued martingale and* (A_i) *is a set-valued predictable increasing sequence such that for all* $\omega \notin B$:

(a) $A_1(\omega) = 0$,

(b) $A_j(\omega) = \overline{\left(\sum_{i=1}^{j-1} \mathcal{E}[F_{i+1}|\Sigma_i](\omega) \ominus F_i(\omega) \right)}$, *for all* $j \geq 2$,

(c) $M_1(\omega) = F_1(\omega)$, *and*

(d) $M_j(\omega) = \overline{\left(\sum_{i=2}^{j} [F_i(\omega) \ominus \mathcal{E}[F_i|\Sigma_{i-1}](\omega) \right)} + F_1(\omega)$, *for all* $j \geq 2$.

Summary

- The main point of this talk is to explain the relationship between collections of subsets of a Banach space and vector spaces with a particular interest in the role of order.
- Near vector spaces and near vector lattices occur frequently in set-valued analysis.

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Thank you for your attention!!