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Some topics on the theory of cones

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Let $X$ be a normed space. A convex subset $P \subseteq X$ is a **cone** in $\lambda P = P$ for any $\lambda \geq 0$. If moreover $P \cap (-P) = \{0\}$, the cone $P$ is **pointed (or proper)**.

Denote $X'$ is the algebraic and $X^*$ topological dual of $X$.

A convex subset $B$ of $P$ is a **base for $P$** if a strictly positive linear functional $f$ of $X$ exists such that

$$B = \{x \in P \mid f(x) = 1\}.$$ 

Then we say that $B$ is **defined by $f$ and is denoted by $B_f$**.

**Theorem 1.** The base $B_f$ of $P$ defined by $f$ is bounded if and only if $f$ is uniformly monotonic (i.e $f(x) \geq a\|x\|$ for each $x \in P$, where $a > 0$ is a real constant).

**Theorem 2.** If $f \in X^*$ is strictly positive we have: The base $B_f$ is bounded if and only if $f$ is an interior point of $P^0$.  
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Unbounded, convex subsets of cones

Suppose that \( \langle X, Y \rangle \) is a dual system \( X, Y \) ordered normed spaces.

For any cone \( P \) of \( X \)

\[
P_Y^0 = \{ y \in Y : \langle x, y \rangle \geq 0 \text{ for each } x \in P \},
\]

is the **dual cone** of \( P \) in \( Y \).

If dual cone of \( X_+ \) in \( Y \) is \( Y_+ \) and the dual cone of \( Y_+ \) in \( X \) is \( X_+ \), \( \langle X, Y \rangle \) is an **ordered dual system**.
For any convex subset $K$ of $P$ and for each $\rho \in \mathbb{R}_+$ we denote

$$K_\rho = \{ x \in K \mid \|x\| \leq \rho \}, \quad K_{S,\rho} = \{ x \in K \mid \|x\| = \rho \},$$

whenever these sets are non-empty.

**Lemma 3. ([4] Lemma 1)** Let $\langle X, Y \rangle$ be an ordered dual system where $X, Y$ are ordered normed spaces and let $K$ be a norm-unbounded, convex subset of the positive cone $X_+$ of $X$.

If the set of quasi-interior positive elements of $Y$ is non-empty, then the following statements are equivalent:

(i) $K_\rho \subseteq \overline{K}_{S,\rho} \sigma(X,Y)$, for each $\rho$,

(ii) $K_\rho \subseteq \overline{\text{co}} \sigma(X,Y) K_{S,\rho}$, for each $\rho$,

(iii) for each $x \in K$ and for each $\rho > \|x\|$, there exists a sequence $\{x_\nu\}$ of $K_{S,\rho}$ which converges to $x$ in the $\sigma(X,Y)$-topology.
Theorem 4. ([7] theorem 4.1) An infinite dimensional Banach lattice $X$ is order isomorphic to $\ell_1(\Gamma)$ if and only if $X$ has the Schur property and $X^*$ has quasi-interior positive elements.
A cone $P$ of a normed space $X$ has the **0-Schur property (or the positive Schur property)** if each weakly convergent to zero sequence of $P$ is norm convergent.

If $P$ has a bounded base $B_f$ with $f \in X^*$, then $P$ has the 0-Schur property, but the converse is not always true.
Theorem 5. ([4] Theorem 10) Suppose that $X$ is a normed space ordered by the pointed cone $P$. Then the following statements are equivalent:

(i) the cone $P$ has a bounded base $B_f$ with $f \in X^*$.

(ii) The cone $P$ has
(a) the 0-Schur property and
(b) the set of quasi-interior positive elements of $X^*$ is non-empty.
Example 6. ([4] Example 11) The space

\[ X = \left( \prod_{n=1}^{\infty} (\mathbb{R}^n)_{\infty} \right)_1, \]

is an ordered Banach space with the Schur property. \( X^* = (\prod_{n=1}^{\infty} (\mathbb{R}^n)_1)_{\infty}. \)

The cone \( X^*_+ \) does not have quasi-interior positive elements, therefore \( X_+ \) does not have a bounded base defined by a continuous linear functional of \( X \). Also 0 is not a point of continuity of \( X_+ \).
A dichotomy result for bases of cones

The cone $P$ is **mixed based** with respect to the family $\mathcal{F}$ of linear functionals of $X$ if $P$ has a bounded and an unbounded base defined (the bases for $P$) by elements of $\mathcal{F}$.

**Theorem 7. ([9] theorem 4)** Suppose that $\langle X, Y \rangle$ is a dual system. If $X$ is a normed space, $P$ is a $\sigma(X, Y)$-closed cone of $X$ so that the positive part $U^+ = U \cap P$ of the unit ball $U$ of $X$ is $\sigma(X, Y)$-compact, then $P$ is not a mixed based cone with respect to the dual cone $P_Y^0$ of $P$ in $Y$. 
**Corollary 1.** Any weak-star closed cone $P$ of the dual $X^*$ of a normed space $X$ is not mixed based with respect to the dual cone $P_0$ of $P$ in $X$.

**Corollary 2.** Any closed cone $P$ of a reflexive Banach space $X$ is not mixed based with respect to the dual cone $P^0$ of $P$ in $X^*$.
Theorem 8. ([2] Lemma 3.4)) Let $X$ be a Banach space ordered by the closed cone $P$. If any base for $P$ defined by a vector of $P^0$ is bounded and at least one such a base exists, then the positive part $U^+ = U \cap P$ of the unit ball $U$ of $X$ is weakly compact.
Definition 9. ([9] ) A normed space $X$ has the property ($\ast$) if for each closed cone $P$ of $X$ we have: either $P$ has no a bounded base defined by an element of $X^*$ or any strictly positive (on $P$) linear functional of $X$ whose restriction on $P$ is continuous in the induced topology of $P$ attains maximum on any base for $P$ which is defined by an element of $X^*$.

It is clear that $X$ has the property ($\ast$) if and only if for any closed cone $P$ of $X$ with a closed, bounded base, any strictly positive (on $P$) linear functional of $X$ whose restriction on $P$ is continuous in the induced topology of $P$ attains maximum on any base for $P$ which is defined by an element of $X^*$.

**Theorem 10.** ([4] Theorem 11) A Banach space $X$ is reflexive if and only if $X$ has the property ($\ast$).

**Theorem 11.** ([2] Theorem 3.6) A Banach space $X$ is reflexive if and only if any closed cone $P$ of $X$ with a closed, bounded base is not mixed based with respect to $P^0$. 
The continuous positive projection property

The continuous positive projection property has been defined in [5]. Let $E$ be a normed space, $P$ a closed cone of $E$ and $x_0 \in P$ an extremal point of $P$ (i.e. $x_0 \neq 0$ and $x \in P, x \leq x_0$ implies $x = tx_0$).

If a continuous, order contractive projection

$$R : E \longrightarrow [x_0]$$

of $E$ onto the one-dimensional subspace generated by $x_0$ exists, such that

$$0 \leq R(x) \leq x \text{ for each } x \in P,$$

then we say that the point $x_0$ has (admits) a continuous, positive projection.

Then a positive, continuous linear functional $r$ of $E$ with $r(x_0) = 1$, exists such that

$$R(x) = r(x)x_0 \text{ for each } x \in E.$$
If each extremal point $x_0 \in P$ (whenever such points exist) admits a continuous positive projection then we say that $E$ has the continuous projection property.
Proposition 12. ([5] Proposition 3.2)

Let $E$ be a normed space ordered by the pointed cone $P$. If

(i) $E$ is a locally solid lattice, or

(ii) $E$ is a Banach space, the cone $P$ is closed and generating and $E$ has the Riesz decomposition property,

then $E$ has the continuous projection property.
Therefore, in many cases, the continuous projection property is weaker than the lattice property and also than the Riesz decomposition property. Indeed in a Banach space ordered by a closed generating pointed cone, the Riesz decomposition property implies the continuous projection property.
Recall also that a continuous linear functional $f$ of $E$ strongly exposes a point $x$ of a subset $D$ of $E$ if $f(x) > f(y)$ for each $y \in D$ and for any sequence $\{x_n\}$ of $D$ we have: $f(x_n) \to f(x)$, implies that $\|x_n - x\| \to 0$. 
Proposition 13. A point $x_0$ of a base $B$ for $P$ is an extreme point of $B$ if and only if $x_0$ is an extremal point of $P$.

Theorem 14. ([5] proposition 3.4) Suppose that $B$ is a base for a cone $P$ of a normed space $E$, defined (the base) by a continuous linear functional $f \in E^*$. If $x_0$ is an extreme point of $B$ which admits a continuous positive projection

$$R(x) = r(x)x_0,$$

we have:

$x_0$ is a strongly exposed point of $B$ if and only if a uniformly monotonic, continuous linear functional $h$ of $E$ exists (i.e. a continuous linear functional $h$ of $E$ exists so that $h(x) \geq a ||x||$ for any $x \in P$, where $a$ is a real constant).

Then the functional

$$g = h(x_0)r - h,$$

strongly exposes $x_0$ in $B$ with $g(x_0) = 0$. 17
Corollary 3. Let $E$ be a Banach space ordered by the closed, generating cone $P$ and suppose that $x_0$ is an extreme point of a base $B$ for $P$.

If $x_0$ admits a continuous positive projection, the following statements are equivalent:

(i) $x_0$ is a strongly exposed point of $B$,

(ii) the cone $P$ has a closed, bounded base.
Conic isomorphisms and a characterization of $\ell_1^+$. 

Suppose that $X, Y$ are normed spaces and $P, Q$, are closed cones of $X, Y$ respectively. We say that the cone $P$ is isomorphic to the cone $Q$ if there exits an one-to-one, map $T$ of $P$ onto $Q$ so that 

$$T(\lambda x + \mu y) = \lambda T(x) + \mu T(y),$$ 

for each $x, y \in P$ and $\lambda, \mu \in \mathbb{R}_+$ and 

$T, T^{-1}$ are continuous,

in the metric topology of $P, Q$ induced by the norm.
By the continuity of $T$ and $T^{-1}$ at zero, there exist real constants $A, B > 0$ such that

$$A||x|| \leq ||T(x)|| \leq B||x||,$$

for any $x \in P$. But this double inequality is not sufficient for the continuity of $T, T^{-1}$ on the whole cone. This is sufficient only for the continuity of $T, T^{-1}$ at zero.

An old result due to Pelzynski Singer and Milman is the following:

**Theorem 15.** (1964) A Banach space $X$ is nonreflexive if and only if $X$ has a closed cone $P$ isomorphic to the positive cone of $\ell_1$. 


Theorem 16. ([6] Proposition 4.1) Let $E$ be a Banach space ordered by the infinite dimensional, closed cone $P$.

Suppose that $E$ has the continuous projection property. If the cone $P$ has the Krein-Milman property, statements (i), (ii), (iii) and (iv) are equivalent. If $P$ has the Radon-Nikodým property, all the following statements are equivalent:

(i) The cone $P$ is isomorphic to $\ell^+_1(\Gamma)$,
(ii) the cone $P$ has a closed, bounded base,
(iii) a base for $P$ defined (the base) by a continuous linear functional, has at least one strongly exposed point,
(iv) the zero is a strongly exposed point of $P$,
(v) the cone $P$ has a dentable base, defined by a continuous linear functional,
(vi) the cone $P$ is dentable,
(vii) each closed and convex subset of $P$ has at least one strongly exposed point.
Theorem 17. ([10] theorem 3) In any Banach space $X$ the following statements are equivalent:

(i) the positive cone $c_0^+$ of $c_0$ is embeddable in $X$,
(ii) the space $c_0$ is embeddable in $X$.

Problem: Give characterizations of $c_0^+$. 
Reflexive cones

Let $X$ be a Banach space and $P$ a cone of $X$.

**Definition 18. ([3])** The cone $P$ is reflexive if the positive part

$$U_+ = U \cap P$$

of the closed unit ball $U$ of $X$ is weakly compact.

Any reflexive cone is closed.
Theorem 19. ([3] Theorem 4.5) The cone $P$ is reflexive if and only if $\ell_1^+$ is not embedddable in $P$.

Theorem 20. ([3] Theorem 3.3) The cone $P$ is reflexive if and only if

$$\hat{P} = P^{00},$$

where $P^{00} = (P^0)^0 \subseteq X^{**}$ is the second dual of $P$.

Theorem 21. ([3] Theorem 3.5)

A Banach space $X$ is reflexive if and only if $X$ has a cone $P$ so that $P$ and $P^0$ are reflexive.
Strongly reflexive cones

**Definition 22.** The cone $P$ is **strong reflexive**, if $U_+ = U \cap P$ is norm compact.

**Theorem 23.** ([3] Theorem 5.6) A Banach space $X$ has the Schur property if and only if any reflexive cone of $X$ is strongly reflexive.
Theorem 24. ([3] Theorem 5.7) If $X$ is a Banach lattice with a positive Schauder basis $\{e_i\}$, then $X_+$ contains a strongly reflexive cone $P$ with a bounded base and $\overline{P - P} = X$.

Example 25. In ([3] Example 5.9) a strongly reflexive cone $P$ of $L_1 [0, 1]$ is given so that

$$P \subseteq L_1^+ [0, 1] \text{ and } \overline{P - P} = L_1 [0, 1].$$
Reflexive cones and compact operators

**Theorem 26. ([3] theorem 5.10)** If $X$ is a Banach space ordered by the strongly reflexive (reflexive) cone $P$, then any positive, linear operator from a Banach lattice $E$ into $X$ is compact (weakly compact).
Reflexive cones and the lattice property

**Theorem 27. ([3] Theorem 7.1)** If $P$ is reflexive and normal cone of a Banach space $X$ then $X$, ordered by the cone $P$, is Dedekind complete.

**Theorem 28. ([3] Theorem 7.3)** If the cone $P$ is reflexive and $P$ has a closed, bounded base $B$, then $P$ does not contain an infinite dimensional, closed cone $K$ with the continuous projection property (hence $P$ does not contain an infinite dimensional, closed cone with the Riesz decomposition property).
The generalized topological dual of a cone and cone isomorphisms.

In this section we define the notion of generalized topological dual of a cone and we give some new results on cone isomorphisms.

Suppose that $E$ is a Banach space $P$ is a closed cone of $X$, $X = P - P$ is the subspace of $E$ generated by $P$ and that $X$ is dense in $E$, i.e. $E = \overline{X}$.

**Definition 29.** Denote by $P^1$ the set of $f : P \rightarrow \mathbb{R}_+$ which are positively homogeneous and additive on $P$ and also are continuous in the metric topology of $P$ defined by the norm of $X$. We call $P'$ the generalized topological dual (or the auto-dual) of $P$. 
If we suppose that $X$ is ordered by the cone $P$ we have

$$P^1 = \{ f \in X_+ | f|_P \text{ is continuous } P \},$$

and

$$X_+^* = P^0 \subseteq P^1 \subseteq X_+.$$
Denote by $F = P^1 - P^1$ be the subspace of $X'$ generated by $P^1$, with norm

$$||f||_F = sup\{|f(x)| \mid x \in U_+\}, \ f \in F.$$ 

Suppose also that $F$ is ordered by the cone $P^1$, i.e. $F_+ = P^1$.

$$U_F^+ = \{f \in F \mid ||f||_P < 1\},$$

is the positive part of the unit ball of $F$.

Denote by $P^2$ the generalized topological dual of $P^1$, i.e.

$$P^2 = (P^1)^1.$$ 

Hence $P^2$ is the set $\phi : P^1 \rightarrow \mathbb{R}_+$ which are additive, positively homogeneous and continuous on $P^1$.
Theorem 30. Suppose that $Y, Z$ are normed spaces and $K, Q$ are closed cones of $Y, Z$ respectively.

If

$$T : K \rightarrow Q,$$

is an isomorphism of $K$ onto $Q$, then

$$T' : Q^1 \rightarrow K^1,$$

is an isomorphism of $Q^1$ onto $K^1$.

In the next theorem denote by $\{e_i \mid i \in \mathbb{N}\}$ the usual Schauder basis of $\ell_1$.

Theorem 31. Suppose that the cone $P$ is separable and normal. If $T$ is an isomorphism of $\ell^+_1$ onto $P^1$ and for any infinite subset $A$ of $\mathbb{N}$ the image $T(K_A)$ of the closed subcone $K_A$ of $\ell^+_1$ generated by the subset $\{e_i \mid i \in A\}$ is not mixed based with respect to the cone $P$, then $E$ contains a copy of $c_0$. 

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Bibliography


