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Some topics on the theory of cones

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Let X be a normed space. A convex subset $P \subseteq X$ is a **cone** in $\lambda P = P$ for any $\lambda \geq 0$. If moreover $P \cap (-P) = \{0\}$, the cone P is **pointed (or proper)**.

Denote X' is the algebraic and X^* topological dual of X.

A convex subset B of P is a **base for** P if a strictly positive linear functional f of X exists such that

$$B = \{x \in P \mid f(x) = 1\}.$$

Then we say that B **is defined by** f **and is denoted by** B_f.

Theorem 1. *The base B_f of P defined by f is bounded if and only if f is uniformly monotonic (i.e $f(x) \geq a\|x\|$ for each $x \in P$, where $a > 0$ is a real constant).*

Theorem 2. *If $f \in X^*$ is strictly positive we have: The base B_f is bounded if and only if f is an interior point of P^0.*

Unbounded, convex subsets of cones

Suppose that $\langle X, Y\rangle$ is a dual system X, Y ordered normed spaces.

For any cone P of X

$$P_Y^0 = \{y \in Y : \langle x, y\rangle \geq 0 \text{ for each } x \in P\},$$

is the **dual cone** of P in Y.

If dual cone of X_+ in Y is Y_+ and the dual cone of Y_+ in X is X_+, $\langle X, Y\rangle$ is an **ordered dual system**.

For any convex subset K of P and for each $\rho \in \mathbf{R}_+$ we denote

$$K_\rho = \{x \in K \mid \|x\| \leq \rho\} \ , \quad K_{S,\rho} = \{x \in K \mid \|x\| = \rho\},$$

whenever these sets are non-empty.

Lemma 3. ([4] Lemma 1) *Let $\langle X, Y \rangle$ be an ordered dual system where X, Y are ordered normed spaces and let K be a norm-unbounded, convex subset of the positive cone X_+ of X.*
If the set of quasi-interior positive elements of Y is non-empty, then the following statements are equivalent:
(i) $K_\rho \subseteq \overline{K_{S,\rho}}^{\sigma(X,Y)}$*, for each* ρ*,*
(ii) $K_\rho \subseteq \overline{\text{co}}^{\sigma(X,Y)} K_{S,\rho}$*, for each* ρ*,*
(iii) *for each $x \in K$ and for each $\rho > \|x\|$, there exists a sequence $\{x_\nu\}$ of $K_{S,\rho}$ which converges to x in the $\sigma(X, Y)$-topology.*

Theorem 4. ([7] theorem 4.1) *An infinite dimensional Banach lattice X is order isomorphic to $\ell_1(\Gamma)$ if and only if X has the Schur property and X^* has quasi-interior positive elements.*

A cone P of a normed space X has the **0-Schur property (or the positive Schur property)** if each weakly convergent to zero sequence of P is norm convergent.

If P has a bounded base B_f with $f \in X^*$, then P has the 0-Schur property, but the converse is not always true.

Theorem 5. ([4] Theorem 10) *Suppose that X is a normed space ordered by the pointed cone P. Then the following statements are equivalent:*

(i) the cone P has a bounded base B_f with $f \in X^*$.

(ii) The cone P has
(a) the 0-Schur property and
(b) the set of quasi-interior positive elements of X^* is non-empty.

Example 6. ([4] Example 11) *The space*

$$X = (\prod_{n=1}^{\infty} (\mathbb{R}^n)_\infty)_1,$$

is an ordered Banach space with the Schur property. $X^* = (\prod_{n=1}^{\infty} (\mathbb{R}^n)_1)_\infty$.

The cone X_+^* *does not have quasi-interior positive elements, therefore* X_+ *does not have a bounded base defined by a continuous linear functional of* X*. Also* 0 *is not a point of continuity of* X_+*.*

A dichotomy result for bases of cones

The cone P is **mixed based** with respect to the family $\mathcal{F}$ of linear functionals of X if P has a bounded and an unbounded base defined (the bases for P) by elements of $\mathcal{F}$.

Theorem 7. ([9] theorem 4) *Suppose that $\langle X, Y \rangle$ is a dual system. If X is a normed space, P is a $\sigma(X,Y)$-closed cone of X so that the positive part $U^+ = U \cap P$ of the unit ball U of X is $\sigma(X,Y)$-compact, then P is not a mixed based cone with respect to the dual cone P_Y^0 of P in Y.*

Corollary 1. *Any weak-star closed cone P of the dual X^* of a normed space X is not mixed based with respect to the dual cone P_0 of P in X.*

Corollary 2. *Any closed cone P of a reflexive Banach space X is not mixed based with respect to the dual cone P^0 of P in X^*.*

Theorem 8. *(**([2] Lemma 3.4)**) Let X be a Banach space ordered by the closed cone P. If any base for P defined by a vector of P^0 is bounded and at least one such a base exists, then the positive part $U^+ = U \cap P$ of the unit ball U of X is weakly compact.*

Definition 9. ([9]) *A normed space X has the property $(*)$ if for each closed cone P of X we have: either P has no a bounded base defined by an element of X^* or any strictly positive (on P) linear functional of X whose restriction on P is continuous in the induced topology of P attains maximum on any base for P which is defined by an element of X^*.*

It is clear that X has the property $(*)$ if and only if for any closed cone P of X with a closed, bounded base, any strictly positive (on P) linear functional of X whose restriction on P is continuous in the induced topology of P attains maximum on any base for P which is defined by an element of X^*.

Theorem 10. ([4] Theorem 11) *A Banach space X is is reflexive if and only if X has the property (*).*

Theorem 11. ([2] Theorem 3.6) *A Banach space X is is reflexive if and only if any closed cone P of X with a closed, bounded base is not mixed based with respect to P^0.*

The continuous positive projection property

The continuous positive projection property has been defined in **[5]**. Let E be a normed space, P a closed cone of E and $x_0 \in P$ an extremal point of P (i.e. $x_0 \neq 0$ and $x \in P, x \leq x_0$ implies $x = tx_0$).

If a continuous, order contractive projection

$$R : E \longrightarrow [x_0]$$

of E onto the one-dimensional subspace generated by x_0 exists, such that

$$0 \leq R(x) \leq x \text{ for each } x \in P,$$

then we say that the point x_0 **has (admits) a continuous, positive projection**.

Then a positive, continuous linear functional r of E with $r(x_0) = 1$, exists such that

$$R(x) = r(x)x_0 \text{ for each } x \in E.$$

If each extremal point $x_0 \in P$ (whenever such points exist) admits a continuous positive projection then we say that E has the **continuous projection property**.

Proposition 12. ([5] Proposition 3.2)

Let E be a normed space ordered by the pointed cone P. If

(i) E is a locally solid lattice, or

(ii) E is a Banach space, the cone P is closed and generating and E has the Riesz decomposition property,

then E has the continuous projection property.

Therefore, in many cases, the continuous projection property is weaker than the lattice property and also than the Riesz decomposition property. Indeed in a Banach space ordered by a closed generating pointed cone, the Riesz decomposition property implies the continuous projection property.

Recall also that a continuous linear functional f of E **strongly exposes** a point x of a subset D of E if $f(x) > f(y)$ for each $y \in D$ and for any sequence $\{x_n\}$ of D we have: $f(x_n) \longrightarrow f(x)$, implies that $\|x_n - x\| \longrightarrow 0$.

Proposition 13. *A point* x_0 *of a base* B *for* P *is an extreme point of* B *if and only if* x_0 *is an extremal point of* P*.*

Theorem 14. ([5] proposition 3.4) *Suppose that* B *is a base for a cone* P *of a normed space* E*, defined (the base) by a continuous linear functional* $f \in E^*$*.*

If x_0 *is an extreme point of* B *which admits a continuous positive projection*

$$R(x) = r(x)x_0,$$

we have:
x_0 *is a strongly exposed point of* B *if and only if a uniformly monotonic, continuous linear functional* h *of* E *exists (i.e. a continuous linear functional* h *of* E *exists so that* $h(x) \geq a||x||$ *for any* $x \in P$*, where* a *is a real constant).*
Then the functional

$$g = h(x_0)r - h,$$

strongly exposes x_0 *in* B *with* $g(x_0) = 0$*.*

Corollary 3. *Let E be a Banach space ordered by the closed, generating cone P and suppose that x_0 is an extreme point of a base B for P.*

If x_0 admits a continuous positive projection, the following statements are equivalent:

(i) x_0 is a strongly exposed point of B,

(ii) the cone P has a closed, bounded base.

Conic isomorphisms and a characterization of ℓ_1^+.

Suppose that $X,\ Y$ are normed spaces and $P,\ Q$, are closed cones of $X,\ Y$ respectively. We say that **the cone P is isomorphic to the cone Q** if there exits an one-to-one, map T of P onto Q so that

$$T(\lambda x + \mu y) = \lambda T(x) + \mu T(y),$$

for each $x,\ y \in P$ and $\lambda, \mu \in \mathbb{R}_+$ and

$$T, T^{-1} \text{ are continuous,}$$

in the metric topology of $P,\ Q$ induced by the norm.

By the continuity of T and T^{-1} at zero, there exist real constants $A, B > 0$ such that

$$A||x|| \leq ||T(x)|| \leq B||x||,$$

for any $x \in P$. But this double inequality is not sufficient for the continuity of T, T^{-1} on the whole cone. This is sufficient only for the the continuity of T, T^{-1} at zero.

An old result due to Pelzynski Singer and Milman is the following:

Theorem 15. *(1964) A Banach space X is nonreflexive if and only if X has a closed cone P isomorphic to positive cone of ℓ_1.*

Theorem 16. ([6] Proposition 4.1) *Let E be a Banach space ordered by the infinite dimensional, closed cone P.*

Suppose that E has the continuous projection property. If the cone P has the Krein-Milman property, statements $(i), (ii), (iii)$ and (iv) are equivalent. If P has the Radon-Nikodým property, all the following statements are equivalent:

(i) *The cone P is isomorphic to $\ell_1^+(\Gamma)$,*

(ii) *the cone P has a closed, bounded base,*

(iii) *a base for P defined (the base) by a continuous linear functional, has at least one strongly exposed point,*

(iv) *the zero is a strongly exposed point of P,*

(v) *the cone P has a dentable base, defined by a continuous linear functional,*

(vi) *the cone P is dentable,*

(vii) *each closed and convex subset of P has at least one strongly exposed point.*

Theorem 17. ([10] theorem 3) *In any Banach space* X *the following statements are equivalent:*
(i) *the positive cone* c_0^+ *of* c_0 *is embeddable in* X*,*
(ii) *the space* c_0 *is embeddable in* X*.*

Problem: Give characterizations of c_0^+.

Reflexive cones

Let X be a Banach space and P a cone of X.

Definition 18. ([3]) *The cone P is* **reflexive** *if the positive part*

$$U_+ = U \cap P$$

of the closed unit ball U of X is weakly compact.

Any reflexive cone is closed.

Theorem 19. ([3] Theorem 4.5) *The cone P is reflexive if and only if ℓ_1^+ is not embedddable in P.*

Theorem 20. ([3] Theorem 3.3) *The cone P is reflexive if and only if*

$$\widehat{P} = P^{00},$$

where $P^{00} = (P^0)^0 \subseteq X^{**}$ is the second dual of P.

Theorem 21. ([3] Theorem 3.5)

A Banach space X is reflexive if and only if X has a cone P so that P and P^0 are reflexive.

Strongly reflexive cones

Definition 22. *The cone P is* **strong reflexive***, if $U_+ = U \cap P$ is norm compact.*

Theorem 23. ([3] Theorem 5.6) *A Banach space X has the Schur property if and only if any reflexive cone of X is strongly reflexive.*

Theorem 24. ([3] Theorem 5.7) *If X is a Banach lattice with a positive Schauder basis $\{e_i\}$, then X_+ contains a strongly reflexive cone P with a bounded base and $\overline{P-P} = X$.*

Example 25. *In* **([3] Example 5.9)** *a strongly reflexive cone P of $L_1[0,1]$ is given so that*

$$P \subseteq L_1^+[0,1] \quad \textit{and} \quad \overline{P-P} = L_1[0,1].$$

Reflexive cones and compact operators

Theorem 26. ([3] theorem 5.10) *If X is a Banach space ordered by the strongly reflexive (reflexive) cone P, then any positive, linear operator from a Banach lattice E into X is compact (weakly compact).*

Reflexive cones and the lattice property

Theorem 27. ([3] Theorem 7.1) *If P is reflexive and normal cone of a Banach space X then X, ordered by the cone P, is Dedekind complete.*

Theorem 28. ([3] Theorem 7.3) *If the cone P is reflexive and P has a closed, bounded base B, then P does not contain an infinite dimensional, closed cone K with the continuous projection property (hence P does not contain an infinite dimensional, closed cone with the Riesz decomposition property).*

The generalized topological dual of a cone and cone isomorphisms.

In this section we define the notion of generalized topological dual of a cone and we give same new results on cone isomorphisms.
Suppose that E is a Banach space P is a closed cone of X,
$X = P - P$ is the subspace of E generated by P and that X is dense in E, i.e. $E = \overline{X}$.
Definition 29. *Denote by* P^1 *the set of* $f : P \longrightarrow \mathbb{R}_+$ *which are positively homogeneous and additive on* P *and also are continuous in the metric topology of* P *defined by the norm of* X*. We call* P' *the* **generalized topological dual (or the auto-dual)** *of* P*.*

If we suppose that X is ordered by the cone P we have

$$P^1 = \{f \in X'_+ \mid f|_P \text{ is continuous } P\},$$

and

$$X^*_+ = P^0 \subseteq P^1 \subseteq X'_+.$$

Denote by $F = P^1 - P^1$ be the subspace of X' generated by P^1, with norm

$$||f||_F = sup\{|f(x)| \mid x \in U_+\},\ f \in F.$$

Suppose also that F is ordered by the cone P^1, i.e. $F_+ = P^1$.

$$U_F^+ = \{f \in F \mid ||f||_P < 1\},$$

is the positive part of the unit ball of F.

Denote by P^2 the generalized topological dual of P^1, i.e.

$$P^2 = (P^1)^1.$$

Hence P^2 is the set $\phi : P^1 \longrightarrow \mathbb{R}_+$ which are additive, positively homogeneous and continuous on P^1

Theorem 30. *Suppose that Y, Z are normed spaces and K, Q are closed cones of Y, Z respectively.*

If

$$T : K \longrightarrow Q,$$

is an isomorphism of K onto Q, then

$$T' : Q^1 \longrightarrow K^1,$$

is an isomorphism of Q^1 onto K^1.

In the next theorem denote by $\{e_i \mid i \in \mathbf{N}\}$ the usual Schauder basis of ℓ_1.

Theorem 31. *Suppose that the cone P is separable and normal. If T is an isomorphism of ℓ_1^+ onto P^1 and for any infinite subset A of $\mathbf{N}$ the image $T(K_A)$ of the closed subcone K_A of ℓ_1^+ generated by the subset $\{e_i \mid i \in A\}$ is not mixed based with respect to the cone P, then E contains a copy of c_0.*

Bibliography

[1] C.D. Aliprantis, R. Tourky, ”Cones and Duality” Graduate Studies in Mathematics, vol. 84, American Mathematical Society, Providence, 2007.

[2] E. Casini, E. Miglierina ”Cones with bounded and unbounded bases and reflexivity” Nonlinear Analysis, vol. 72 (2010), 2356-2366.

[3] E. Casini, E. Miglierina, I.A. Polyrakis and F. Xanthos ”Reflexive Cones” ”Positivity vol. 17 nr.3 (2013) 911-933.

[4] C. Kountzakis and I. A. Polyrakis ”Geometry of cones and an application to the theory of Parerto efficient points”, Journal of Math. Anal and Appl.”, vol. 320 (2006) 340-351.

[5] I. A. Polyrakis, ”Strongly exposed points in bases for the positive cone of ordered Banach spaces and characterizations of $\ell_1(\Gamma)$ ”, Proc. Edinb. Math. Society ”, vol.29 (1986) 271-282.

[6] I. A. Polyrakis”, ”Cones locally isomorphic to the positive cone of $\ell_1(\Gamma)$”, ”Linear Algebra and its Applications”, vol.84 (1986) 323-334.

[7] I. A. Polyrakis, ”Strongly exposed points and a characterization of $\ell_1(\Gamma)$ by the Schur Property”, Proc. Edinb. Math. Society, vol. 30 (1987 397-400.

[8] I. A. Polyrakis, ”The Radon-Nikodým Property in Ordered Banach Spaces ”, Journal of Math. Analysis and Applications, vol. 192 (1995) 381-391.

[9] I.A. Polyrakis ”Demand functions and reflexivity”,

Journal of Math. Anal and Appl.”, vol. 338 (2008) 695-704.

[10] I.A. Polyrakis and F. Xanthos ”Cone characterization of Grothendieck spaces and Banach spaces containing c_0”, ”Positivity vol. 15 (2011) 677-693.