Generalized Lorentz spaces and Köthe duality

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Outline

1. Generalized Lorentz spaces
2. Köthe duality
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Lorentz and Orlicz-Lorentz spaces

Let $I = (0, a)$ be a finite or infinite interval, $m$ the Lebesgue measure and $w$ a positive weight on $I$.

Recall

For $0 < p < \infty$ the classical Lorentz space $\Lambda_{p,w}$ is defined by

$$\Lambda_{p,w} = \{ f \in L_0(I) : \int_I (f^*)^p w \ dm < \infty \}$$

where $f^*$ is the decreasing rearrangement of $f$.

Similarly if $\varphi$ is an Orlicz function, the Orlicz-Lorentz space $\Lambda_{\varphi,w}$ is defined by

$$\Lambda_{\varphi,w} = \{ f \in L_0(I) : \exists c > 0, \int_I \varphi(cf^*) w \ dm < \infty \}$$
Both Lorentz and Orlicz-Lorentz spaces are defined on the same pattern: Given a symmetric function space $E$, (e.g. $E = L^p$ or $E = L_\varphi$), define a weighted version $E_w$ of $E$ (which is no more symmetric) and then take the symmetrization of $E_w$ defined by

$$f \in (E_w)^{\text{sym}} \iff f^* \in E_w$$

Next we define properly the weighted space $E_w$. 
Desymmetrized function spaces

Let $w$ be a positive weight over $I = (0, a)$, $a \in (0, \infty]$. We set $\omega = w \cdot m$ ($m =$Lebesgue measure) and $b = \omega(I) = \int_I w \, dm$. Let $E$ be a symmetric Banach or quasi-Banach function space over $J = (0, b)$. For $f \in L_0(J)$ let $f_w^*$ be its positive decreasing rearrangement with respect to the measure $\omega$.

**Definition**

The desymmetrized space $E_w$ of $E$ by $w$ is defined by

$$f \in E_w \iff f_w^* \in E$$

with (quasi-) norm

$$\| f \|_{E_w} = \| f_w^* \|_E$$

The space $E_w$ is an order isometric realization of $E$ as Banach function space over the interval $I$. It is symmetric with respect to the measure $\omega$ but not w.r. to the Lebesgue measure.
Generalized Lorentz spaces associated with a desymmetrized function space

Let as before $w$ be a positive weight on $I$, $E$ a symmetric function space on $J$ and $E_w$ the corresponding desymmetrized space.

**Definition**

We denote by $\Lambda_{E,w}$ the symmetrization of $E_w$, defined by

$$f \in \Lambda_{E,w} \iff f^* \in E_w$$

where $f^*$ is the ordinary decreasing rearrangement of $f$ (i.e. with respect to the Lebesgue measure). We call $\Lambda_{E,w}$ a generalized Lorentz space.

The class $\Lambda_{E,w}$ may be trivial. In fact

$$\Lambda_{E,w} \neq (0) \iff 1 \in E \text{ or } \exists t > 0 : \int_0^t w \, dm < \infty$$
Conditions on the weight

It is clear that
\[ \forall t \in I, \left( \chi_{[0,t]} \right)_w^* = \chi_{[0,W(t)]} \]

where \( W(t) = \int_0^t w \, dm \). It is natural to assume \( W(t) < \infty \) for all \( t < \infty \). Then \( W \) is an increasing homeomorphism from \( I \) onto \( J \). Moreover

\[ \forall f \in L_0(I), \forall \lambda > 0, \quad \omega(|f| > \lambda) = m(|f| \circ W^{-1} > \lambda) \]

Consequently

\[ f_w^* = (f \circ W^{-1})^* \]

If \( W \) verifies condition \( \Delta_2 \), then \( \Lambda_{E,w} \) is a quasi-Banach symmetric function space with quasi-norm

\[ \| f \|_{E,w} = \| f^* \|_{E_w} = \| f^* \circ W^{-1} \|_E \]
The case $w$ decreasing

The class $\Lambda_{E,w}$ need not to be a Banach function space when $E$ is (even if $W$ has $\Delta_2$). However if $w$ is non-increasing then this is the case. More precisely:

**Proposition**

Assume that $w$ is non-increasing and that $E$ is a strongly symmetric Banach function space. Then $\Lambda_{E,w}$ is also a strongly symmetric Banach function space.

**Recall**

$f \prec g$ ($f$ is submajorized by $g$) if $\int_0^t f^* dm \leq \int_0^t g^* dm$ for all $t > 0$. $E$ is strongly symmetric if $\forall f, g \in E, f \prec g \implies \|f\|_E \leq \|g\|_E$. 
Proof: Since $W(t)/t$ is nonincreasing, the function $W$ verifies $\Delta_2$; it follows that $\Lambda_{E,w}$ is a quasi-Banach function space, with quasi-norm $\|f\|_{E,w} = \|f^* \circ W^{-1}\|_E$. Now to prove the triangular inequality in $\Lambda_{E,w}$ it will be sufficient to prove the submajorization inequality

$$(f + g)^* \circ W^{-1} \prec f^* \circ W^{-1} + g^* \circ W^{-1}$$

for every $f, g \in \Lambda_{E,w}$. Indeed since $E$ is assumed to be strongly symmetric it will follow that

$$\|(f + g)^* \circ W^{-1}\|_E \leq \|(f^* + g^*) \circ W^{-1}\|_E$$

$$\leq \|(f^* \circ W^{-1}\|_E + \|g^* \circ W^{-1}\|_E$$

But

$$\int_0^t (f + g)^* \circ W^{-1} \, dm = \int_0^{W^{-1}(t)} (f + g)^* w \, dm$$

$$\leq \int_0^{W^{-1}(t)} (f^* + g^*) w \, dm = \int_0^t (f^* + g^*) \circ W^{-1} \, dm$$

(by $f + g \prec f^* + g^*$ and the fact that $w$ is nonincreasing). \qed
**Duality of \( E_w \)**

**Fact**

Let \( E \) be a symmetric Banach function space, \( E' \) its Köthe dual, \( w \) be a positive measurable weight. Then \((E_w)' = w.E'_w\), which means

\[ g \in (E_w)' \iff g/w \in (E')_w \text{ with } \|g\|_{E_w} = \|g/w\|_{(E')_w} \]

Indeed we have

\[ g \in E'_w \iff \forall f \in E_w, \; gf \in L^1(w.m) \]
\[ g \in (E_w)' \iff \forall f \in E_w, \; gf \in L^1(m) \]

**Definition**

We denote by \( M_{E,w} \) the symmetrization of the Banach function space \( w.E_w \). We have thus

\[ g \in M_{E,w} \iff (g^*/w)^* \in E \]

In the case where \( W \) is finite everywhere, then

\[ g \in M_{E,w} \iff (g^*/w) \circ W^{-1} \in E \]
Structure of $M_{E,w}$

$M_{E,w}$ is a solid, starlike, rearrangement invariant subset of $L_0(I)$. It contains indicator functions of sets of finite measure. It is not necessarily closed by addition. It is equipped with a distinguished homogeneous faithful functional $\| M \|

\[ \| f \|_M = \| f^*/w \|_{E_w} = \| (f^*/w) \circ W^{-1} \|_E \]

If $M_{E,w}$ is closed by addition the functional $\| M \|$ is a quasi-norm. This happens in particular when $1/w$ has $\Delta_2$ property.
Duality of $M_{E,w}$

By Köthe dual of a subset $A$ of $L_0(I, m)$ (which need not to be a linear subspace) we mean the set

$$A' = \{ g \in L_0(I) : \forall f \in A, \ gf \in L_1(I, m) \}$$

**Proposition**

Assume that $w$ is a non-increasing weight with $W(t) < \infty$ for finite $t$. Then the Köthe dual of $M_{E,w}$ is $\Lambda_{E',w}$. 

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Generalized Lorentz spaces
Normability of $M_{E,w}$

We give an (only sufficient) criterion for the class $M_{E,w}$ ($w$ non-increasing) to be normable.

**Definition**

A non increasing weight on the interval $I$ is called *regular* if

$$\exists C < \infty, \forall t \in I, W(t) \leq Ctw(t)$$

(note that the converse inequality $tw(t) \leq W(t)$ is always true).

**Theorem**

Assume that $w$ is non-increasing and regular, and that $E$ is strongly symmetric. Then the class $M_{E,w}$ is a normable vector lattice.

In fact the functional $\|\cdot\|_{M_{E,w}}$ is $C$-equivalent to a lattice norm, where $C$ is the regularity constant of $w$. 
Application to the duality of $\Lambda_{E,w}$

**Corollary**

If $E$ is a symmetric space with Fatou property and $w$ a regular nonincreasing weight then the Köthe dual of $\Lambda_{E,w}$ is $M_{E\prime,w}$.

Indeed in this case we have

$$\Lambda'_{E,w} = M''_{E\prime,w} = M_{E\prime,w}$$
The spaces $\Lambda_{E,w}$, with $w > 0$ nonincreasing may be described as an intersection of desymmetrized spaces $E_v$. More precisely

**Proposition**

For every $f \in \Lambda_{E,w}$ it holds that $\|f\|_{E,w} = \sup \{ \|f\|_{E_v} : v \geq 0, v^* = w \}$.

In fact for every $v \geq 0, v^* = w$ we have $f^*_v \prec (f^*)_w^*$. Indeed

\[
\int_0^t f^*_v dm = \inf_{\lambda > 0} \left( \int_I (|f| - \lambda)_+ v dm + \lambda t \right) \\
\leq \inf_{\lambda > 0} \left( \int_I ((|f| - \lambda)_+)^* w dm + \lambda t \right) \\
= \inf_{\lambda > 0} \left( \int_I (f^* - \lambda)_+ w dm + \lambda t \right) = \int_0^t (f^*)_w^* dm
\]
Inequalities for rearrangements and weights: classes $M_{E,w}$

**Proposition**

Assume $w$ is nonincreasing and $E$ is a strongly symmetric B.f.s. Then for every $f \in M_{E,w}$ it holds that

$$\|f\|_{M_{E,w}} = \inf \{ \|f_v\|_E : v \geq 0, v^* = w, \text{supp } v \supseteq \text{supp } f \}.$$ 

In fact for every $v \geq 0, v^* = w$ with $\text{supp } v \supseteq \text{supp } f$ we have

$$\left(\frac{f^*}{w}\right)_w^* \prec \left(\frac{f}{v}\right)_v^*.$$ 

If $W(t) < \infty$ for $t$ finite, this may be written

$$\left(\frac{f^* \circ W^{-1}}{w}\right)^* \prec \left(\frac{f}{v} \circ V^{-1}\right)^*$$

when $v > 0$ (or for $v \geq 0$ with an appropriate definition of $V^{-1}$).
Example

Let $A_1, \ldots, A_n$ be disjoint sets with $m(A_1) \geq m(A_2) \geq \cdots \geq m(A_n)$ and $\lambda_1, \lambda_2, \ldots, \lambda_N$ be real numbers. Then

$$\left\| \sum_{i=1}^{n} \frac{\lambda_i^*}{m(A_i)} \chi_{A_i} \right\|_E \leq \left\| \sum_{i=1}^{n} \frac{\lambda_i}{m(A_i)} \chi_{A_i} \right\|_E$$

Proof: Set $w_i = m(A_i)$, $w = \sum_{i=1}^{n} w_i \chi_{[i,i+1)}$, $f = \sum_{i=1}^{n} \frac{\lambda_i}{m(A_i)} \chi_{[i,i+1)}$ and write down the inequality $\| f \|_{M(E,w)} \leq \| f \|_{w,E_w}$ (that is, $\|(f^*/w) \circ W^{-1}\|_E \leq \|(f/v) \circ V^{-1}\|_E)$.

$\square$
References


