Generalized Lorentz spaces and Köthe duality

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Outline

Lorentz and Orlicz-Lorentz spaces

Let $I = (0, a)$ be a finite or infinite interval, m the Lebesgue measure and w a positive weight on I.

Recall

For $0 < p < \infty$ the classical Lorentz space $\Lambda_{p,w}$ is defined by

$$\Lambda_{p,w} = \{f \in L_0(I) : \int_I (f^*)^p w \, dm < \infty\}$$

where f^* is the decreasing rearrangement of f.

Similarly if φ is an Orlicz function, the Orlicz-Lorentz space $\Lambda_{\varphi,w}$ is defined by

$$\Lambda_{\varphi,w} = \{f \in L_0(I) : \exists c > 0, \int_I \varphi(cf^*) w \, dm < \infty\}$$

Both Lorentz and Orlicz-Lorentz spaces are defined on the same pattern: Given a symmetric function space E, (e.g. $E = L_p$ or $E = L_\varphi$), define a weighted version E_w of E (which is no more symmetric) and then take the symmetrization of E_w defined by

$$f \in (E_w)^{\mathrm{s}ym} \iff f^* \in E_w$$

Next we define properly the weighted space E_w.

Desymmetrized function spaces

Let w be a positive weight over $I = (0, a)$, $a \in (0, \infty]$.
We set $\omega = w \cdot m$ (m=Lebesgue measure) and $b = \omega(I) = \int_I w\, dm$.
Let E be a symmetric Banach or quasi-Banach function space over $J = (0, b)$.
For $f \in L_0(J)$ let f_w^* be its positive decreasing rearrangement with respect to the measure ω.

Definition

The desymmetrized space E_w of E by w is defined by

$$f \in E_w \iff f_w^* \in E$$

with (quasi-) norm

$$\|f\|_{E_w} = \|f_w^*\|_E$$

The space E_w is an order isometric realization of E as Banach function space over the interval I. It is symmetric with respect to the measure ω but not w.r. to the Lebesgue measure.

Generalized Lorentz spaces associated with a desymmetrized function space

Let as before w be a positive weight on I, E a symmetric function space on J and E_w the corresponding desymmetrized space.

Definition

We denote by $\Lambda_{E,w}$ the symmetrization of E_w, defined by

$$f \in \Lambda_{E,w} \iff f^* \in E_w$$

where f^* is the ordinary decreasing rearrangement of f (i. e. with respect to the Lebesgue measure). We call $\Lambda_{E,w}$ a generalized Lorentz space.

The class $\Lambda_{E,w}$ may be trivial. In fact

$$\Lambda_{E,w} \neq (0) \iff 1 \in E \text{ or } \exists t > 0 : \int_0^t w\, dm < \infty$$

Conditions on the weight

It is clear that

$$\forall t \in I, \left(\chi_{[0,t]}\right)^*_w = \chi_{[0,W(t)]}$$

where $W(t) = \int_0^t w\, dm$. It is natural to assume $W(t) < \infty$ for all $t < \infty$. Then W is an increasing homeomorphism from I onto J. Moreover

$$\forall f \in L_0(I), \forall \lambda > 0, \quad \omega(|f| > \lambda) = m(|f| \circ W^{-1} > \lambda)$$

Consequently

$$f^*_w = (f \circ W^{-1})^*$$

If W verifies condition Δ_2, then $\Lambda_{E,w}$ is a quasi-Banach symmetric function space with quasi-norm

$$\|f\|_{E,w} = \|f^*\|_{E_w} = \|f^* \circ W^{-1}\|_E$$

The case w decreasing

The class $\Lambda_{E,w}$ need *not* to be a *Banach* function space when E is (even if W has Δ_2).
However if w is non-increasing then this is the case. More precisely:

Proposition

Assume that w is non-increasing and that E is a strongly symmetric Banach function space. Then $\Lambda_{E,w}$ is also a strongly symmetric Banach function space.

Recall

$f \prec g$ (f is *submajorized* by g) if $\int_0^t f^*\, dm \leq \int_0^t g^* dm$ for all $t > 0$.
E is *strongly symmetric* if $\forall f, g \in E$, $f \prec g \implies \|f\|_E \leq \|g\|_E$.

Proof: Since $W(t)/t$ is nonincreasing, the function W verifies Δ_2; it follows that $\Lambda_{E,w}$ is a quasi-Banach function space, with quasi-norm $\|f\|_{E,w} = \|f^* \circ W^{-1}\|_E$. Now to prove the triangular inequality in $\Lambda_{E,w}$ it will be sufficient to prove the submajorization inequality

$$(f+g)^* \circ W^{-1} \prec f^* \circ W^{-1} + g^* \circ W^{-1}$$

for every $f, g \in \Lambda_{E,w}$. Indeed since E is assumed to be strongly symmetric it will follow that

$$\begin{aligned}\|(f+g)^* \circ W^{-1}\|_E &\leq \|(f^* + g^*) \circ W^{-1}\|_E \\ &\leq \|(f^* \circ W^{-1}\|_E + \|g^* \circ W^{-1}\|_E\end{aligned}$$

But

$$\begin{aligned}\int_0^t (f+g)^* \circ W^{-1}\, dm &= \int_0^{W^{-1}(t)} (f+g)^* w\, dm \\ &\leq \int_0^{W^{-1}(t)} (f^* + g^*) w\, dm = \int_0^t (f^* + g^*) \circ W^{-1}\, dm\end{aligned}$$

(by $f + g \prec f^* + g^*$ and the fact that w is nonincreasing). □

Duality of E_w

Fact

Let E be a symmetric Banach function space, E' its Köthe dual, w be a positive measurable weight. Then $(E_w)' = w.E'_w$, which means

$$g \in (E_w)' \iff g/w \in (E')_w \text{ with } \|g\|_{E_w} = \|g/w\|_{(E')_w}$$

Indeed we have

$$g \in E'_w \iff \forall f \in E_w,\ gf \in L_1(w.m)$$
$$g \in (E_w)' \iff \forall f \in E_w,\ gf \in L_1(m)$$

Definition

We denote by $M_{E,w}$ the symmetrization of the Banach function space $w.E_w$. We have thus

$$g \in M_{E,w} \iff (g^*/w)^*_w \in E$$

In the case where W is finite everywhere, then

$$g \in M_{E,w} \iff (g^*/w) \circ W^{-1} \in E$$

Structure of $M_{E,w}$

$M_{E,w}$ is a solid, starlike, rearrangement invariant subset of $L_0(I)$.
It contains indicator functions of sets of finite measure.
It is not necessarily closed by addition.
It is equipped with a distinguished homogeneous faithful functional $\| \, \|_M$

$$\|f\|_M = \|f^*/w\|_{E_w} = \|(f^*/w) \circ W^{-1}\|_E$$

If $M_{E,w}$ is closed by addition the functional $\| \, \|_M$ is a quasi-norm. This happens in particular when $1/w$ has Δ_2 property.

Duality of $M_{E,w}$

By Köthe dual of a subset A of $L_0(I, m)$ (which need not to be a linear subspace) we mean the set

$$A' = \{g \in L_0(I) : \forall f \in A,\ gf \in L_1(I, m)\}$$

Proposition

Assume that w is a non-increasing weight with $W(t) < \infty$ for finite t. Then the Köthe dual of $M_{E,w}$ is $\Lambda_{E',w}$.

Normability of $M_{E,w}$

We give an (only sufficient) criterion for the class $M_{E,w}$ (w non-increasing) to be normable.

Definition

A non increasing weight on the interval I is called *regular* if

$$\exists C < \infty, \forall t \in I, W(t) \leq Ctw(t)$$

(note that the converse inequality $tw(t) \leq W(t)$ is always true).

Theorem

Assume that w is nonincreasing and regular, and that E is strongly symmetric. Then the class $M_{E,w}$ is a normable vector lattice.

In fact the functional $\|.\|_{M_{E,w}}$ is C-equivalent to a lattice norm, where C is the regularity constant of w.

Application to the duality of $\Lambda_{E,w}$

Corollary

If E is a symmetric space with Fatou property and w a regular nonincreasing weight then the Köthe dual of $\Lambda_{E,w}$ is $M_{E',w}$.

Indeed in this case we have

$$\Lambda'_{E,w} = M''_{E',w} = M_{E',w}$$

Inequalities for rearrangements and weights: spaces $\Lambda_{E,w}$

The spaces $\Lambda_{E,w}$, with $w > 0$ nonincreasing may be described as an intersection of desymmetrized spaces E_v. More precisely

Proposition

For every $f \in \Lambda_{E,w}$ it holds that $\|f\|_{E,w} = \sup\{\|f\|_{E_v} : v \geq 0, v^* = w\}$.

In fact for every $v \geq 0, v^* = w$ we have $f_v^* \prec (f^*)_w^*$. Indeed

$$\begin{aligned}\int_0^t f_v^* dm &= \inf_{\lambda>0} \Big(\int_I (|f| - \lambda)_+ v \, dm + \lambda t \Big) \\ &\leq \inf_{\lambda>0} \Big(\int_I ((|f| - \lambda)_+)^* w \, dm + \lambda t \Big) \\ &= \inf_{\lambda>0} \Big(\int_I (f^* - \lambda)_+ w \, dm + \lambda t \Big) = \int_0^t (f^*)_w^* dm\end{aligned}$$

Inequalities for rearrangements and weights: classes $M_{E,w}$

Proposition

Assume w is nonincreasing and E is a strongly symmetric B.f.s. Then for every $f \in M_{E,w}$ it holds that
$\|f\|_{M_{E,w}} = \inf\{\|\frac{f}{v}\|_{E_v} : v \geq 0, v^* = w, \operatorname{supp} v \supseteq \operatorname{supp} f\}$.

In fact for every $v \geq 0, v^* = w$ with $\operatorname{supp} v \supseteq \operatorname{supp} f$ we have

$$\left(\frac{f^*}{w}\right)^*_w \prec \left(\frac{f}{v}\right)^*_v$$

If $W(t) < \infty$ for t finite, this may be written

$$\left(\frac{f^*}{w} \circ W^{-1}\right)^* \prec \left(\frac{f}{v} \circ V^{-1}\right)^*$$

when $v > 0$ (or for $v \geq 0$ with an appropriate definition of V^{-1}).

Example

Let $A_1, \ldots, A_n$ be disjoint sets with $m(A_1) \geq m(A_2) \geq \cdots \geq m(A_n)$ and $\lambda_1, \lambda_2, \ldots \lambda_N$ be real numbers. Then

$$\left\| \sum_{i=1}^{n} \frac{\lambda_i^*}{m(A_i)} \chi_{A_i} \right\|_E \leq \left\| \sum_{i=1}^{n} \frac{\lambda_i}{m(A_i)} \chi_{A_i} \right\|_E$$

Proof: Set $w_i = m(A_i)$, $w = \sum_{i=1}^{n} w_i \chi_{[i,i+1)}$, $f = \sum_{i=1}^{n} \frac{\lambda_i}{m(A_i)} \chi_{[i,i+1)}$
and write down the inequality $\|f\|_{M(E,w} \leq \|f\|_{w.E_w}$
(that is, $\|(f^*/w) \circ W^{-1}\|E \leq \|(f/v) \circ V^{-1}\|_E$). □

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