HIGHER ORDER MONOTONIC FUNCTIONS OF SEVERAL VARIABLES

Paul Ressel

Katholische Universität Eichstätt-Ingolstadt

1. Introduction

For an interval $I \subseteq \mathbb{R}$ and $f : I \longrightarrow \mathbb{R}$ we consider

$$(\Delta_h f)(s) := f(s+h) - f(s)$$

$(h > 0; s, s+h \in I)$, $\Delta_h^2 := \Delta_h \circ \Delta_h$ etc., and call f *n-absolutely monotone* if

$$\Delta^p f \geq 0 \qquad \text{for } p = 1, \ldots, n \quad \text{(where defined).}$$

Note that $f \geq 0$ is not assumed.
Let $A_1, \ldots, A_n \subseteq \overline{\mathbb{R}}$ be non-empty, $A := A_1 \times \ldots \times A_n$, and let $\varphi : A \longrightarrow \mathbb{R}$ be any function.
Then for $a = (a_1, \ldots, a_n), b = (b_1, \ldots, b_n) \in A$ we put

$$\begin{aligned} D_a^b \varphi \quad := \quad & \varphi(b) - \varphi(a_1, b_2, \ldots, b_n) - \ldots - \varphi(b_1, \ldots, b_{n-1}, a_n) \\ + \quad & \varphi(a_1, a_2, b_3, \ldots, b_n) + \varphi(a_1, b_2, a_3, b_4, \ldots, b_n) \\ + \quad & \ldots + \varphi(b_1, \ldots, b_{n-2}, a_{n-1}, a_n) \\ - \quad & \varphi(a_1, a_2, a_3, b_4, \ldots, b_n) - \ldots + - \ldots + (-1)^n \varphi(a) \,. \end{aligned}$$

Definition.

φ is *n-increasing* iff $D_a^b \varphi \geq 0 \quad \forall\, a < b$ in A ; φ is called *fully n-increasing* iff φ with k of the variables fixed is $(n-k)$-increasing in the remaining variables, for every choice of these variables, and for every $k = 0, 1, \ldots, n-1$. Here $a < b$ means $a_j < b_j$ for all $j = 1, \ldots, n$.

Theorem 1 (Correspondence Theorem).

Suppose $\sup A_i \in A_i$ *for alle* $i \leq n$. *Then* $\varphi : A \longrightarrow \mathbb{R}_+$ *is fully n-increasing and right continuous iff*

$$\varphi(a) = \mu([-\infty, a] \cap \overline{A}) \qquad , \quad a \in A$$

for some (unique) $\mu \in M_+(\overline{A})$, $\overline{A}$ *denoting the closure in* $\overline{\mathbb{R}}^n$.

That is, φ is the *distribution function* (d.f.) of μ .

Note that here A_i is not assumed to be an interval.

There is a fundamental connection between the two types of higher order monotonicity mentioned so far:

Theorem 2.

$f : I \longrightarrow \mathbb{R}$ is n-absolutely monotone if and only if $f \circ \varphi$ is fully n-increasing for each fully n-increasing φ with values in I.

On $I = [0,1]$ we can say a little more: let $f : I \longrightarrow I$ be continuous in 1 with value $f(1) = 1$. Then there are equivalent:

- **(i)** *f is n-absolutely monotone*
- **(ii)** *For every n-dimensional d.f. F of some probability measure on $\mathbb{R}^n$ also $f \circ F$ is a d.f..*
- **(iii)** $[0,1]^n \ni x \longmapsto f\left(\frac{1}{n}\sum_{i=1}^{n} x_i\right)$ *is a d.f..*

A natural question arises: if F is an m-dimensional d.f., G an n-dimensional d.f., for which functions f on $[0, 1]^2$ is $f \circ (F \times G)$ again a d.f.?

Until recently the answer was known only for $m = n = 1 : f$ has to be a 2-dimensional d.f., and a particularly important class of such functions on $[0, 1]^2$ are the bivariate copulas. For $m, n \geq 1$ the function f has to fulfill stronger (multivariate) monotonicity conditions, and these will now be discussed.

Let $I_1, \ldots, I_d$ be any non-degenerate intervals in $\mathbb{R}$, $I := I_1 \times \ldots \times I_d$, and let $f : I \longrightarrow \mathbb{R}$ be any function. For $s \in I, h \in [0, \infty[^d$ such that also $s + h \in I$ put

$$(E_h f)(s) := f(s+h)$$

and $\Delta_h := E_h - E_0$, i.e. $(\Delta_h f)(s) := f(s+h) - f(s) =: -(\nabla_h f)(s)$.

Since the family of operators $\{E_h\}$ is commutative (where defined), so is the family $\{\Delta_h\}$. In particular (with $e_1, \ldots, e_d$ denoting standard unit vectors in $\mathbb{R}^d$), $\Delta_{h_1 e_1}, \ldots, \Delta_{h_d e_d}$ commute.

Definition. Let $\boldsymbol{n} = (n_1, \ldots, n_d) \in \mathbb{N}^d$. Then $f : I \longrightarrow \mathbb{R}$ is called $\boldsymbol{n}$-$\uparrow$ (read: $\boldsymbol{n}$-*increasing*) iff

$$\left(\Delta_h^{\boldsymbol{p}} f\right)(s) := \left(\Delta_{h_1 e_1}^{p_1} \Delta_{h_2 e_2}^{p_2} \ldots \Delta_{h_d e_d}^{p_d} f\right)(s) \geq 0$$

for all $s \in I, h = (h_1, \ldots, h_d) \in]0, \infty[^d$, $\boldsymbol{p} = (p_1, \ldots, p_d) \in \mathbb{N}_0^d, 0 \lneqq \boldsymbol{p} \leq \boldsymbol{n}$ such that $s_j + p_j h_j \in I_j \ \forall j \leq d$. If instead

$$\left(\nabla_h^{\boldsymbol{p}} f\right)(s) := \left(\nabla_{h_1 e_1}^{p_1} \nabla_{h_2 e_2}^{p_2} \dots \nabla_{h_d e_d}^{p_d} f\right)(s) \geq 0 \,,$$

f is called $\boldsymbol{n}$-$\downarrow$ (read: $\boldsymbol{n}$-*decreasing*).
We'll say that f is $\boldsymbol{p}$ times (continuously) differentiable if

$$f_{\boldsymbol{p}} := \frac{\partial^{|\boldsymbol{p}|} f}{\partial s_1^{p_1} \dots \partial s_d^{p_d}} \qquad \left(|\boldsymbol{p}| := \sum_{i=1}^{d} p_i\right)$$

exists (and is continuous). We shall use the same symbol also in cases where only the right partial derivative(s) exist. If f is $\boldsymbol{n}$ times differentiable, then

$$f \text{ is } \boldsymbol{n}\text{-}\uparrow \iff f_{\boldsymbol{p}} \geq 0 \quad \forall\, 0 \lneqq \boldsymbol{p} \leq \boldsymbol{n}$$

and

$$f \text{ is } \boldsymbol{n}\text{-}\downarrow \iff (-1)^{|\boldsymbol{p}|} f_{\boldsymbol{p}} \geq 0 \quad \forall\, 0 \lneqq \boldsymbol{p} \leq \boldsymbol{n} \,.$$

We shall in the following mostly consider the intervals $[0,1], [0,1[$ or $\mathbb{R}_+ = [0,\infty[$, thereby certainly not restricting the generality. Since

$$\left(\Delta_h^{\mathbf{1}_d} f\right)(s) = D_s^{s+h} f \quad , \qquad \left[\mathbf{1}_d := (1,1,\ldots,1) \in \mathbb{N}^d\right]$$

we see that being fully d-increasing is the same as being $\mathbf{1}_d$ -$\uparrow$. The property $\boldsymbol{n}$ -$\uparrow$ for some $\boldsymbol{n} \geq \mathbf{1}_d$ thus corresponds to a stronger monotonicity requirement, beyond being just a distribution function.

If f is $\boldsymbol{n}$-$\uparrow$ then obviously f considered as a function of only one variable s_i is n_i-$\uparrow$ for each i. The converse however is not true: $f(s_1, s_2) := (s_1 s_2 - a)_+$ with $a > 0$ is 2-$\uparrow$ as a function of s_1 or s_2. But

$$\left(\Delta_{(h,h)}^{(2,2)} f\right)(0) = 4(h^2 - a)_+ - 4(2h^2 - a)_+ + (4h^2 - a)_+$$

has the value $-a$ for $h = \sqrt{a}$, showing f to be not $(2,2)$-$\uparrow$. That f is $(1,1)$-$\uparrow$ can be seen directly, and is also a consequence of Theorem 1. The function f is not even $(1,2)$-$\uparrow$, since

$$\left(\Delta_{he_2}^2 f\right)(s_1, s_2) = (s_1(s_2+2h)-a)_+ - 2(s_1(s_2+h)-a)_+ + (s_1 s_2 - a)_+$$

is not increasing in s_1: for $a = s_1 = s_2 = h = 1$ this expression is 0, and changing s_1 to $\frac{1}{2}$ yields the value $\frac{1}{2}$.

Theorem 3.

Let $n \geq 2, f : [0,1[\longrightarrow \mathbb{R}_+$. *Then* f *is* n-$\uparrow$ *if and only if there exist uniquely determined* $a_0, \ldots, a_{n-2} \geq 0$ *and a measure* μ *on* $[0,1[$ *such that*

$$f(t) = a_0 + a_1\, t + \ldots + a_{n-2}\, t^{n-2} + \int (t-a)_+^{n-1}\, d\mu(a), \quad 0 \leq t < 1 .$$

The function f *is continuous and for* $n > 2$ $(n-2)$ *times continuously differentiable on* $[0,1[$, *where* $f^{(m)}$ *is* $(n-m)$-$\uparrow$, $m = 1, \ldots, n-2$. *The right derivative of* $f^{(n-2)}$ *exists and equals* $(n-1)! \cdot \mu([0,t])$, *and is therefore right-continuous and increasing; in particular* $\mu(\{0\}) = f_r^{(n-1)}(0)/(n-1)!$. *The constants* a_j *are given by* $a_j = f^{(j)}(0)/j!$, $j \leq n-2$.

Our proof is based on some elementary facts about convex functions, and some arguments used by Widder for *absolutely monotone* functions (i.e. n-absolutely monotone for all $n \in \mathbb{N}$).

We introduce functions $f_a, 0 \leq a \leq 1$, on $[0,1]$:

$$f_a(t) := \frac{(t-a)_+}{1-a} \text{ for } 0 \leq a < 1, \quad f_1 := 1_{\{1\}} =: f_0^\infty$$

(note that $f_0(t) = t$). Furthermore

$$K_n := \{f : [0,1] \longrightarrow \mathbb{R}_+ \mid f \text{ is } n\text{-}\uparrow, f(1) = 1\} .$$

As a relatively easy consequence of Theorem 2 we obtain

Theorem 4.

K_n is a Bauer simplex (for $n \geq 2$), and

$$ex(K_n) = \left\{ f_0^j \mid j = 0, 1, \ldots, n-2 \right\} \cup \{ f_a^{n-1} \mid a \in [0,1] \} .$$

A function $f \in K_n$ is continuous on $[0,1[$, and in 1 if and only if the measure associated doesn't charge the singleton $\{f_1\}$.

The extreme points of K_n were already identified in 1964 by *McLachlan* (Pac. J. Math. **14**), in a direct but rather complicated way. That K_n is a simplex was not shown there.

Functions which are n-$\uparrow$ for every $n \in \mathbb{N}$ have been known for a long time already, they are called *absolutely monotone*. We might abbreviate this property by „∞-$\uparrow$“. Let $K_\infty := \{f : [0,1] \longrightarrow \mathbb{R}_+ \mid f \text{ is } \infty\text{-}\uparrow, f(1) = 1\}$. Then we may state the following properties, due essentially to Widder:

Theorem 5.

(i) *K_∞ is a Bauer simplex, and $ex(K_\infty) = \{f_0^j \mid j \in \overline{\mathbb{N}}_0\}$.*

(ii) *$f : [0,1[\longrightarrow \mathbb{R}_+$ is absolutely monotone iff f is analytic with non-negative coefficients.*

There are counterparts for n-$\downarrow$ functions. We consider the direct one

$$L_n := \{g : [0,1] \longrightarrow \mathbb{R}_+ \mid g \text{ is } n\text{-}\downarrow,\ g(0) = 1\}$$

and also

$$H_n := \{g : \mathbb{R}_+ \longrightarrow \mathbb{R}_+ \mid g \text{ is } n\text{-}\downarrow,\ g(0) = 1\}\,.$$

Theorem 6.

For $n \geq 2$ both L_n and H_n are Bauer simplices. With $g_c(s) := (1 - cs)_+$, for $c \in [0, \infty]$ (where $g_\infty := 1_{\{0\}}$) we have

$$ex(L_n) = \left\{g_1^j \mid j = 0, \ldots, n-2\right\} \cup \{g_c^{n-1} \mid c \in [1, \infty]\}$$

$$ex(H_n) = \{g_c^{n-1} \mid c \in [0, \infty]\}\,.$$

Here the structure of H_n goes back to Schoenberg and Williamson.

3. The multivariate case

If $F \subseteq \mathbb{R}^X, G \subseteq \mathbb{R}^Y$ for some sets X, Y, then

$$F \otimes G := \{f \otimes g \mid f \in F, g \in G\},$$

not the linear space generated by these functions.

Put $E_n := \mathrm{ex}(K_n) = \{f_0^j \mid j = 0, \ldots, n-2\} \cup \{f_a^{n-1} \mid a \in [0,1]\}$.
Note that $f_1 = f_1^{n-1}$ is the only discontinuous function in E_n.
$A_n := E_n \setminus \{f_1\}$ will also be important.
For $\boldsymbol{n} = (n_1, \ldots, n_d) \in (\mathbb{N} \setminus 1)^d$ we define

$$\begin{aligned} E_{\boldsymbol{n}} &:= E_{n_1} \otimes \ldots \otimes E_{n_d}, \\ A_{\boldsymbol{n}} &:= A_{n_1} \otimes \ldots \otimes A_{n_d}, \end{aligned}$$

and in analogy to dimension 1

$$K_{\boldsymbol{n}} := \left\{ f : [0,1]^d \longrightarrow \mathbb{R}_+ \mid f \text{ is } \boldsymbol{n}\text{-}\uparrow, f(\mathbf{1}_d) = 1 \right\} ,$$

obviously a compact and convex set of functions.
Our main result is the following

Theorem 7.

$E_{\boldsymbol{n}}$ is a Bauer simplex, and $ex(K_{\boldsymbol{n}}) = E_{\boldsymbol{n}}$ ($\boldsymbol{n} \geq 2_d$).

We will give a sketch of the proof, consisting of five steps.

I. Let $f \in K_{\boldsymbol{n}}$ be given and assume f is C^∞ . Assume first $\boldsymbol{n} = 2_d$. The partial derivative f_{1_d} fulfills $(f_{1_d})_{1_d} = f_{2_d} \geq 0$, hence $f_{1_d}(t) = \nu([0,t])$ for some $\nu \in M_+([0,1]^d)$ by the Correspondence Theorem, yielding

$$
\begin{aligned}
D^s_{\mathbf{0}_d} f &= \int_{[0,s]} f_{\mathbf{1}_d}(t)\,dt = \int_{[0,s]} \nu([0,t])\,dt \\
&= \int_{[0,s]} \int_{[0,1]^d} 1_{[0,t]}(a)\,d\nu(a)\,dt \\
&= \int_{[0,1]^d} \lambda\!\!\lambda^d([0,s] \cap [a, \mathbf{1}_d])\,d\nu(a) \\
&= \int_{[0,1]^d} \prod_{i=1}^{d} (s_i - a_i)_+\,d\nu(a) = \int_{[0,1]^d} \prod_{i=1}^{d} f_{a_i}(s_i)\,d\tilde{\nu}(a)
\end{aligned}
$$

where $\tilde{\nu}$ is a finite measure on $[0,1]^d$, concentrated on $[0,1[^d$.

For $\emptyset \neq \alpha \subseteq \underline{d} := \{1, \ldots, d\}$ let $f_\alpha(s_\alpha) := f(s_\alpha, 0_{\alpha^c})$, then

$$f(s) = \sum_{\emptyset \neq \alpha \subseteq \underline{d}} D_{\mathbf{0}_\alpha}^{s_\alpha} f_\alpha + f(\mathbf{0}_d) ,$$

and for every non-empty $\alpha \subseteq \underline{d}$ there is a finite measure $\tilde{\nu}_\alpha$ on $[0, 1[^\alpha$ such that

$$D_{\mathbf{0}_\alpha}^{s_\alpha} f_\alpha = \int_{[0,1[^\alpha} \prod_{i \in \alpha} f_{a_i}(s_i)\, d\tilde{\nu}_\alpha(a_\alpha) \quad , \quad s_\alpha \in [0, 1]^\alpha .$$

Since $s_i = 0$ for $i \in \alpha^c$ in $D_{\mathbf{0}_\alpha}^{s_\alpha} f_\alpha$, we can write (with $\eta := f_0^0 \equiv 1$)

$$D_{\mathbf{0}_\alpha}^{s_\alpha} f_\alpha = \int \prod_{i=1}^{d} h_i(s_i)\, d\left(\tilde{\nu}_\alpha \otimes \varepsilon_{\boldsymbol{\eta}_{\alpha^c}}\right)(h_1, \ldots, h_d) .$$

The measure

$$\mu := \tilde{\nu} + \sum_{\emptyset \neq \alpha \subsetneq \boldsymbol{d}} \tilde{\nu}_\alpha \otimes \varepsilon_{\boldsymbol{\eta}_{\alpha^c}} + f(\mathbf{0}_d) \cdot \varepsilon_{\boldsymbol{\eta}_d}$$

on $E_{\mathbf{2}_d}$ then fulfills

$$f(s) = \int h(s)\, d\mu(h) \quad \forall\, s \in [0,1]^d \,,$$

and it is a probability measure because of

$$1 = f(\mathbf{1}_d) = \sum_{\emptyset \neq \alpha \subseteq \underline{d}} D_{\mathbf{0}_\alpha}^{\mathbf{1}_\alpha} f_\alpha + f(\mathbf{0}_d) = \mu([0,1]^d) \,.$$

Assuming the result to hold for some $\boldsymbol{n} \geq 2_d$ it can then be deduced for $(n_1 + 1, n_2, \ldots, n_d)$, etc. ... .

II. In the second step we only assume $f \in K_{\boldsymbol{n}}$ to be continuous. We now make use of Bernstein polynomials, defined in dimension one by

$$b_{i,k}(t) := \binom{k}{i} t^i (1-t)^{k-i} \quad , \quad i = 0, \ldots, k \, ,$$

and in higher dimensions by

$$B_{\boldsymbol{i},k} := b_{i_1,k} \otimes \ldots \otimes b_{i_d,k} \quad , \quad \boldsymbol{i} = (i_1, \ldots, i_d) \in \{0, \ldots k\}^d \, .$$

It is well known that the Bernstein approximations

$$f_k := \sum_{\mathbf{0}_d \leq \boldsymbol{i} \leq \boldsymbol{k}_d} f\left(\frac{\boldsymbol{i}}{k}\right) \cdot B_{\boldsymbol{i},k}$$

converge to f (even uniformly) on $[0,1]^d$. Applying the formula for derivatives of one-dimensional Bernstein approximations d times, we get

$$(f_k)_{\boldsymbol{p}} = c_{\boldsymbol{p}} \sum_{\boldsymbol{i} \leq \boldsymbol{k}_d - p} \Delta^{p_1}_{e_1/k} \ldots \Delta^{p_d}_{e_d/k} f\left(\frac{\boldsymbol{i}}{k}\right) b_{i_1,k-p_1} \otimes \ldots \otimes b_{i_d,k-p_d}$$

with $c_{\boldsymbol{p}} := \prod_{i=1}^{d} k(k-1) \cdot \ldots \cdot (k - p_i + 1)$; hence $(f_k)_{\boldsymbol{p}} \geq 0$ for $0 \lneqq \boldsymbol{p} \leq \boldsymbol{n}$. Applying the first part of this proof to f_k we have

$$f_k(s) = \int h(s)\, d\mu_k(h) \quad , \quad s \in [0,1]^d ,$$

for suitable $\mu_1, \mu_2, \ldots \in M^1_+(E_{\boldsymbol{n}})$, and with a limit point μ of some convergent subsequence of $\{\mu_1, \mu_2, \ldots\}$ we get

$$f(s) = \int h(s)\, d\mu(h) \quad , \quad s \in [0,1]^d .$$

Already at this point the uniqueness of the integral representation (for continuous f) can and has to be established.

III. In the third step we shall consider a function f defined only on $[0,1[^d$ and being $\boldsymbol{n}$-$\uparrow$. We show first that f is necessarily continuous. Let $s \in [0,1[^d$ be given and consider

$$g(r) := f(s + r \cdot \mathbf{1}_d) \quad \text{for} \quad 0 \leq r < \min_{1 \leq i \leq d} (1 - s_i) \, .$$

g can be shown to be 2-$\uparrow$ and hence is continuous. But the function f is in particular (simply) increasing, and this property combined with the continuity of g (for every choice of s) shows that f is (everywhere on $[0,1[^d$) continuous.
We can now apply part II. to the restriction of f to $[0,c]^d$, for $0 < c < 1$, i.e.

$$f(cs) = \int_{E_{\boldsymbol{n}}} h(s) \, d\nu_c(h) \quad \text{for} \quad s \in [0,1]^d \, ,$$

where $\nu_c \in M_+(E_{\boldsymbol{n}})$ is unique, and is concentrated on $A_{\boldsymbol{n}}$. This can be rewritten with some $\mu_c \in M_+(A_{\boldsymbol{n}})$ as

$$f(u) = \int_{A_{\boldsymbol{n}}} h\left(\frac{u}{c}\right)\, d\nu_c(h) = \int_{A_{\boldsymbol{n}}} h(u)\, d\mu_c(h) \quad , \quad u \in [0,c]^d ,$$

and μ_c turns out to be concentrated on the compact subset $A_c := \prod_{i=1}^{d} \left(\{0,\ldots,n_i-2\} \,\dot\cup\, [0,c]\right)$ of $A_{\boldsymbol{n}} = \prod_{i=1}^{d} \left(\{0,\ldots,n_i-2\} \,\dot\cup\, [0,1[\right)$. Since μ_c on A_c is determined by f , we have for $0 < c_1 < c_2 < 1$ the identity $\mu_{c_2} \mid A_{c_1} = \mu_{c_1}$. Hence for $0 < c_k \nearrow 1$ the measures μ_{c_k} are compatible and so determine a Radon measure μ on $A_{\boldsymbol{n}}$ for which

$$f(s) = \int_{A_{\boldsymbol{n}}} h(s)\, d\mu(h) \quad , \quad s \in [0,1[^d .$$

Here the unicity of μ is an immediate consequence of the way we showed its existence.

IV. Whereas $\boldsymbol{n}$-$\uparrow$ functions on $[0,1[^d$ are automatically continuous, and smooth to a certain degree $(n_i \geq 2 \;\forall\, i)$, this need not be the case for such functions on $[0,1]^d$, as we saw already for $d = 1$. In higher dimensions the situation is a lot more involved, since on each part $T_\alpha := \left\{ s \in [0,1]^d \mid s_i < 1 \Longleftrightarrow i \in \alpha \right\}$ of the „upper right boundary" $[0,1]^d \setminus [0,1[^d,\ \emptyset \neq \alpha \subsetneqq \underline{d}$, one may add „independently" functions of $|\alpha|$ variables with sufficiently high monotonicity. For a given $\boldsymbol{n}$-$\uparrow$ function f on $[0,1]^d$ this „procedure" has to be reversed, i.e. these different parts of the function have to be identified.

This part of the proof is rather involved and I'll omit its details.

V. Here the uniqueness of the representation is proved in the general case. Let $f \in K_{\boldsymbol{n}}$ be represented as

$$f(s) = \int_{E_{\boldsymbol{n}}} h(s)\, d\nu(h) \quad , \quad s \in [0,1]^d \ ;$$

we have to show that $\nu = \mu$, the measure found before. We introduce the parts

$$\begin{aligned} Q_\alpha &:= \{h = (h_1, \ldots, h_d) \in E_{\boldsymbol{n}} \mid h_i = \vartheta \Longleftrightarrow i \in \alpha^c\} \\ &= \prod_{i \in \alpha} A_{n_i} \times \{\boldsymbol{\vartheta}_{\alpha^c}\} \end{aligned}$$

of $E_{\boldsymbol{n}}$ for $\alpha \subseteq \underline{d}$, where $Q_\emptyset = \{\boldsymbol{\vartheta}_d\}$ and $Q_{\underline{d}} = A_{\boldsymbol{n}}$.
The restriction $\nu|Q_\alpha$, i.e. the measure $B \longmapsto \nu(B \cap Q_\alpha)$, has for $\emptyset \neq \alpha \subsetneqq \underline{d}$ the form $\nu_\alpha \otimes \varepsilon_{\boldsymbol{\vartheta}_{\alpha^c}}$ for some $\nu_\alpha \in M_+(A_{\boldsymbol{n}_\alpha})$, $\nu|Q_\emptyset = c' \cdot \varepsilon_{\boldsymbol{\vartheta}_d}$ for some $c' \geq 0$, and $\nu|Q_{\underline{d}} =: \nu_{\underline{d}} \in M_+(A_{\boldsymbol{n}})$.

The (restricted) unicity from III. shows $\nu_{\underline{d}} = \mu_{\underline{d}}$, and then a „backward induction“ (over the size $|\alpha|$ of $\alpha \subseteq \underline{d}$) shows finally $\nu_\alpha = \mu_\alpha \; \forall \; \alpha \subseteq \underline{d}$, i.e. $\mu = \nu$. $\square$

Corollary 8.

If $f \in K_{\boldsymbol{n}}$ is continuous at 1_d , it is every where continuous.

Part III of the proof just given is in fact a multivariate analogue of Theorem 2:

Theorem 9.

Let $\boldsymbol{n} \geq \boldsymbol{2}_d$, $f : [0,1[^d \longrightarrow \mathbb{R}_+$ *. Then* f *is* $\boldsymbol{n}$*-*$\uparrow$ *iff*

$$f(s) = \int h(s)\, d\mu(h) \quad , \quad s \in [0,1[^d ,$$

for a uniquely determined Radon measure μ *on* $A_{\boldsymbol{n}}$ *. The function* f *is continuous and* $(\boldsymbol{n} - \boldsymbol{2}_d)$ *times continuously differentiable. For* $\boldsymbol{m} \leq \boldsymbol{n} - \boldsymbol{2}_d$ *the derivative* $f_{\boldsymbol{m}}$ *is* $(\boldsymbol{n} - \boldsymbol{m})$*-*$\uparrow$ *. The right derivative of* $f_{\boldsymbol{n}-\boldsymbol{2}_d}$ *with respect to each variable exists, is given by* $(\boldsymbol{n} - \boldsymbol{1}_d)! \cdot \mu([0,s])$ *, and is therefore still right-continuous and* $\boldsymbol{1}_d$*-*$\uparrow$ *. For* $\boldsymbol{j} \leq \boldsymbol{n} - \boldsymbol{2}_d$ *we have*

$$\mu\left(\left\{f_0^{j_1} \otimes \ldots \otimes f_0^{j_d}\right\}\right) = f_{\boldsymbol{j}}(\boldsymbol{0}_d)/\boldsymbol{j}!$$

and this holds also if $j_i = n_i - 1$ *for some (or all)* $i \leq d$ *,* $f_{\boldsymbol{j}}$ *denoting then the right derivative in those coordinates.*

If a function of d variables is $\boldsymbol{n}$-$\uparrow$ for every $\boldsymbol{n} \in \mathbb{N}^d$, we call it again *absolutely monotone*.

Theorem 10.

(i) $f : [0,1[^d \longrightarrow \mathbb{R}_+$ *is absolutely monotone if and only if* f *is analytic with non-negative coefficients.*

(ii) K_{∞_d} *is a Bauer simplex, and*
$ex(K_{\infty_d}) = \{f_0^{i_1} \otimes \ldots \otimes f_0^{i_d} \mid i_1, \ldots, i_d \in \overline{\mathbb{N}}_0\}$.

4. COMPOSITION OF HIGHER ORDER MONOTONIC FUNCTIONS

We have the following multivariate generalisation of Theorem 2:

Theorem 11.

Let $f : [0,1]^d \longrightarrow \mathbb{R}_+$ *and* $\boldsymbol{n} \in \mathbb{N}^d$ *be given. Then there are equivalent:*

(i) f *is* $\boldsymbol{n}$-$\uparrow$

(ii) *For any fully* n_i*-increasing* $\varphi_i : \{0,1\}^{n_i} \longrightarrow [0,1]$, $1 \le i \le d$, *the composition* $f \circ (\varphi_1 \times \cdots \times \varphi_d)$ *is fully* $|\boldsymbol{n}|$*-increasing on* $\{0,1\}^{|\boldsymbol{n}|}$.

We indicate the proof of „(i) $\Longrightarrow$ (ii)“ for $\boldsymbol{n} \geq \mathbf{2}_d$. Since f is $\boldsymbol{n}$-$\uparrow$ it has by Theorem 7 (our main result) the representation

$$f(s) = \int \prod_{i=1}^{d} h_i(s_i)\, d\mu(h_1, \ldots, h_d) \quad , \quad s \in [0,1]^d ,$$

for some measure μ on $E_{\boldsymbol{n}}$. This leads to

$$f \circ (\varphi_1 \times \ldots \times \varphi_d) = \int \bigotimes_{i=1}^{d} (h_i \circ \varphi_i)\, d\mu(h_1, \ldots, h_d) ,$$

where each $h_i \circ \varphi_i$ is $\mathbf{1}_{n_i}$-$\uparrow$ by Theorem 2, hence $\bigotimes_{i=1}^{d}(h_i \circ \varphi_i)$ is $1_{|\boldsymbol{n}|}$-$\uparrow$, and so is $f \circ (\varphi_1 \times \ldots \times \varphi_d)$ as a mixture of these functions.

We close with our second main result:

Theorem 12.

Given $\boldsymbol{n} = (n_1, \ldots, n_d) \in \mathbb{N}^d, \boldsymbol{m}_i \in \mathbb{N}^{n_i}, i \le d$, *put* $\ell_i := |\boldsymbol{m}_i|$, $\boldsymbol{\ell} := (\ell_1, \ldots, \ell_d)$ *and* $\boldsymbol{m} := (\boldsymbol{m}_1, \ldots, \boldsymbol{m}_d) \in \mathbb{N}^{|\boldsymbol{n}|}$ *. Let* $g_i : [0,1]^{n_i} \longrightarrow [0,1]$ *be* $\boldsymbol{m}_i$-$\uparrow$, $i \le i \le d$ *, and suppose* $f : [0,1]^d \longrightarrow \mathbb{R}$ *to be* $\boldsymbol{\ell}$-$\uparrow$ *, then* $f \circ (g_1 \times \ldots \times g_d)$ *is* $\boldsymbol{m}$-$\uparrow$ *.*

In dimension $d = 2$ this means:

If $g_1 \times g_2 : [0,1]^{n_1} \times [0,1]^{n_2} \longrightarrow [0,1]^2 \xrightarrow{f} \mathbb{R}$, where g_i is $\boldsymbol{m}_i$-$\uparrow$, $i = 1, 2$, and f is $(|\boldsymbol{m}_1|, |\boldsymbol{m}_2|)$-$\uparrow$, then $f \circ (g_1 \times g_2)$ is $(\boldsymbol{m}_1, \boldsymbol{m}_2)$-$\uparrow$.

The special case $\boldsymbol{m}_i = 1_{n_i}$ (hence $|\boldsymbol{m}_i| = n_i$) answers our question from the beginning:

> Precisely the (n_1, n_2)-$\uparrow$ functions f on $[0,1]^2$ lead to $\mathbf{1}_{n_1+n_2}$-$\uparrow$ compositions $f \circ (g_1 \times g_2)$, i.e. to new distribution functions.