DOMINATION CONDITIONS FOR FAMILIES OF QUASINEARLY SUBHARMONIC FUNCTIONS AND SOME RELATED PROBLEMS

JUHANI RIIHENTAUS
Department of Mathematical Sciences, University of Oulu
P.O. Box 3000, 90014 Oulun Yliopisto, Finland
juhani.riihentaus@gmail.com

1. RESULTS OF DOMAR AND RIPPON

Let $D$ be a domain of $\mathbb{R}^n$, $n \geq 2$. Let $F: D \to [0, +\infty]$ be an upper semicontinuous function – observe that the value $+\infty$ is allowed! Let $\mathcal{F}$ be a family of subharmonic functions $u: D \to [-\infty, +\infty)$ which satisfy

$$u(x) \leq F(x)$$

for all $x \in D$. Write

$$w(x) := \sup_{u \in \mathcal{F}} u(x), \ x \in D,$$

and let $w^*: D \to [-\infty, +\infty]$ be the upper semicontinuous regularization of $w$, that is

$$w^*(x) := \limsup_{y \to x} w(y).$$

Domar [2, Theorem 1 and Theorem 2, pp. 430-431] gave the following result:

**Theorem A.** If for some $\varepsilon > 0$,

$$\int_D [\log^+ F(x)]^{n-1+\varepsilon} \, dm_n(x) < +\infty,$$

then $w$ is locally bounded above in $D$, and thus $w^*$ is subharmonic in $D$.

Rippon [7, Theorem 1, p. 128] generalized Domar’s result in the following form:

**Theorem B.** Let $\varphi: [0, +\infty] \to [0, +\infty]$ be an increasing function such that

$$\int_1^{+\infty} \frac{dt}{[\varphi(t)]^{1/(n-1)}} < +\infty.$$

If

$$\int_D \varphi(\log^+ F(x)) \, dm_n(x) < +\infty,$$

then $w$ is locally bounded above in $D$, and thus $w^*$ is subharmonic in $D$.

As pointed out by Domar, [2, pp. 436-440], and by Rippon, [7, p. 129], the above results are for many particular cases sharp. As Domar also points out, [2, p. 430], the result of his theorem holds in fact for more general functions, that is, for functions which by good reasons might be – and indeed already have been! – called quasinearly subharmonic functions. See below for the definition of this function class.

Date: September 3, 2013.
2. SUBHARMONIC FUNCTIONS AND GENERALIZATIONS

We recall that an upper semicontinuous function \( u : D \rightarrow [-\infty, +\infty) \) is subharmonic if for all closed balls \( B^n(x,r) \subset D, \)

\[
u \leq \frac{1}{\nu_n} \int_{B^n(x,r)} u(y) \, dm(y).
u \]

The function \( u \equiv -\infty \) is considered subharmonic.

We say that a function \( u : D \rightarrow [-\infty, +\infty) \) is nearly subharmonic, if \( u \) is Lebesgue measurable, \( u^+ \in L^1_{\text{loc}}(D), \) and for all \( B^n(x,r) \subset D, \)

\[
u \leq \frac{1}{\nu_n} \int_{B^n(x,r)} u(y) \, dm(y).
u \]

Observe that in the standard definition of nearly subharmonic functions one uses the slightly stronger assumption that \( u \in L^1_{\text{loc}}(D), \) see e.g. [3, p. 14]. However, our above, slightly more general definition seems to be more practical.

We say that a Lebesgue measurable function \( u : D \rightarrow [-\infty, +\infty) \) is \( K \)-quasinearly subharmonic, if \( u^+ \in L^1_{\text{loc}}(D) \) and if there is a constant \( K = K(n,u,D) \geq 1 \) such that for all \( B^n(x,r) \subset D, \)

\[
u \leq \frac{1}{\nu_n} \int_{B^n(x,r)} u_M(y) \, dm(y) \]

for all \( M \geq 0, \) where \( u_M := \max\{u, -M\} + M. \) A function \( u : D \rightarrow [-\infty, +\infty) \) is quasinearly subharmonic, if \( u \) is \( K \)-quasinearly subharmonic for some \( K \geq 1. \)

Observe that a function \( u \) is \( 1 \)-quasinearly subharmonic if and only if it is nearly subharmonic (in the above slightly more general sense).

We recall here only that this function class includes, among others, subharmonic functions, and, more generally, quasiharmonic and nearly subharmonic functions (see e.g. [3, pp. 14, 26]), also functions satisfying certain natural growth conditions, especially certain eigenfunctions, and polyharmonic functions. Also, the class of Harnack functions is included, thus, among others, nonnegative harmonic functions as well as nonnegative solutions of some elliptic equations. In particular, the partial differential equations associated with quasiregular mappings belong to this family of elliptic equations.

3. AN IMPROVEMENT TO THE RESULTS OF DOMAR AND RIPPON

Our improvement to the results of Domar and Rippon is the following:

3.1. Theorem. Let \( K \geq 1. \) Let \( \varphi : [0, +\infty) \rightarrow [0, +\infty] \) and \( \psi : [0, +\infty) \rightarrow [0, +\infty] \) be increasing functions for which there are \( s_0, s_1 \in \mathbb{N}, \) \( s_0 < s_1, \) such that

(i) the inverse functions \( \varphi^{-1} \) and \( \psi^{-1} \) are defined on \( [\min\{\varphi(s_1 - s_0), \psi(s_1 - s_0)\}, +\infty], \)

(ii) \( 2K(\psi^{-1} \circ \varphi)(s - s_0) \leq (\psi^{-1} \circ \varphi)(s) \) for all \( s \geq s_1. \)

(iii) the following series is convergent:

\[
\sum_{j=s_1+1}^{+\infty} \left[ \frac{(\psi^{-1} \circ \varphi)(j+1)}{(\psi^{-1} \circ \varphi)(j)} \cdot \frac{1}{\varphi(j-s_0)} \right]^{1/(n-1)} < +\infty.
\]
Let $\mathcal{F}_K$ be a family of $K$-quasinearly subharmonic functions $u : D \to [-\infty, +\infty)$ such that $u(x) \leq F_K(x)$ for all $x \in D$, where $F_K : D \to [0, +\infty]$ is a Lebesgue measurable function. If for each compact set $E \subset D$,

$$\int_E \psi(F_K(x)) \, dm_n(x) < +\infty,$$

then the family $\mathcal{F}_K$ is locally (uniformly) bounded in $D$. Moreover, the function $w^* : D \to [0, +\infty)$ is a $K$-quasinearly subharmonic function. Here

$$w^*(x) := \limsup_{y \to x} w(y),$$

where

$$w(x) := \sup_{u \in \mathcal{F}_K} u^+(x).$$

Remark. Our theorem is indeed flexible. One corollary is obtained by replacing the above condition (iii) e.g. by

(iii’) The following integral is convergent:

$$\int_{s_1}^{+\infty} \frac{(\psi^{-1} \circ \phi)(s + 2)}{(\psi^{-1} \circ \phi)(s)} \cdot \frac{1}{\phi(s - s_0)} \, ds < +\infty.$$

Another corollary is the following:

3.2. Corollary. Let $K \geq 1$. Let $\phi : [0, +\infty) \to [0, +\infty]$ and $\phi : [0, +\infty) \to [0, +\infty]$ be strictly increasing functions for which there are $s_0, s_1 \in \mathbb{N}$, $s_0 < s_1$, such that

(i) $\phi^{-1} : [0, +\infty) \to [0, +\infty]$ satisfies the $\Delta_2$-condition,

$$\phi^{-1}(2s) \leq C\phi^{-1}(s)$$

for all $s \geq 0$,

(ii) $2K\phi^{-1}(e^{s_0}) \leq \phi^{-1}(e^s)$ for all $s \geq s_1$,

(iii) the following integral is convergent:

$$\int_{s_1}^{+\infty} \frac{ds}{\phi(s - s_0)^{1/(n-1)}} < +\infty.$$

Let $\mathcal{F}_K$ be a family of $K$-quasinearly subharmonic functions $u : D \to [-\infty, +\infty)$ such that $u(x) \leq F_K(x)$ for all $x \in D$, where $F_K : D \to [0, +\infty]$ is a Lebesgue measurable function. If for each compact set $E \subset D$,

$$\int_E \phi(\log^+ \phi(F_K(x))) \, dm_n(x) < +\infty,$$

then the family $\mathcal{F}_K$ is locally (uniformly) bounded in $D$. Moreover, the function $w^* : D \to [0, +\infty)$ is a $K$-quasinearly subharmonic function. Here

$$w^*(x) := \limsup_{y \to x} w(y),$$
where  
\[ w(x) := \sup_{u \in \mathcal{F}_K} u^+(x). \]

**Remark.** To get a more concrete corollaries, choose for example \( p > 0 \) and \( \phi(t) = t^p \). The special case \( p = 1 \) and \( K = 1 \) then gives Domar’s and Rippon’s results. A more complicated, but still a concrete corollary, is obtained by choosing \( p > 0, q > 0 \) and \( \phi(t) = \frac{t^p}{|\log t|^q} \), say.

4. **Separately Subharmonic Functions**

For results on this area, see [1, 4, 5] and the references therein. Here we state, as an example, only the following partial result:

4.1. **Theorem.** Let \( \Omega \) be a domain in \( \mathbb{R}^{m+n} \), \( m, n \geq 2 \). Let \( u : \Omega \rightarrow (-\infty, +\infty) \) be such that

(a) for each \( y \in \mathbb{R}^n \) the function  
\[ \Omega(y) \ni x \mapsto u(x, y) \in [-\infty, +\infty) \]  
is nearly subharmonic, and, for almost every \( y \in \mathbb{R}^n \), subharmonic,

(b) for each \( x \in \mathbb{R}^m \) the function  
\[ \Omega(x) \ni y \mapsto u(x, y) \in [-\infty, +\infty) \]  
is upper semicontinuous, and, for almost every \( x \in \mathbb{R}^m \), (nearly) subharmonic,

(c) for some \( p > 0 \) there is a function \( v \in L^p_{\text{loc}}(\Omega) \) such that \( u \leq v \).

Then \( u \) is upper semicontinuous and thus subharmonic in \( \Omega \).

**Remark.** Observe that the above result is partially related to the result [3, Proposition 2, pp. 34–35]: Though our assumptions are slightly stronger, our proof is, on the other hand, different and shorter.

**References**


