Integration in complex Riesz space setting and some application in harmonic analysis

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The problem of recovering the coefficients in the classical case:

$$\sum_{n=-\infty}^{+\infty} a_n e^{2\pi inx} = f(x)$$

$$\Rightarrow a_n(f) = \frac{1}{\pi}(M)\int_T f(x)e^{-2\pi inx}dx.$$

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In the locally compact case, if the integral

$$\int_{-\infty}^{\infty} a(x) e^{ixy} dx,$$

with $a(x)$ locally summable, converges to a function $f(y)$, the problem of recovering is to recover the function $a(x)$, i.e., to obtain an inversion formula. For a locally summable f it was

$$a(x) = (C,1)\text{-}\lim_{h\to\infty} \frac{1}{2\pi} \int_{-h}^{h} f(y) e^{-ixy} dy$$

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By this we extend our results from

A. Boccuto, V. Skvortsov

Henstock-Kurzweil type integration of Riesz-space-valued functions and applications to Walsh series, Real Analysis Exchange Vol 29 (2003/2004), 419-439.

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So for $h = f + ig$ we define:

$$|h| := sup_\theta(f\cos\theta + g\sin\theta).$$

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The subgroups G_n are clopen w.r. to this topology. Under our assumption G_n/G_{n+1} is finite for each n. So G_n (and all its cosets) is compact.

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Let the order of the factor group G_n/G_{n+1} be p_n. Then the order of G_0/G_1 is p_0, and the order of G_0/G_n, $n = 1, 2, \ldots$, is $m_n := p_0 \cdot p_1 \cdot \ldots \cdot p_{n-1}$, with $p_i \geq 2$ for all i (put $m_0 := 1$).

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Denote by K_n any coset of G_n and by $K_n(g)$ the coset

$$K_n(g) = g + G_n.$$

For each $g \in G$ the uniquely defined sequence $\{K_n(g)\}$ is decreasing and $\{g\} = \cap_n K_n(g)$.

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Under the assumptions imposed on G the group Γ is also a periodic locally compact zero-dimensional abelian group (with respect to the pointwise multiplication of characters). It can be represented as a sum of increasing sequence of subgroups:

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introducing a topology in Γ.
Then $\Gamma = \cup_{i=-\infty}^{+\infty} \Gamma_i$ and $\cap_{i=-\infty}^{+\infty} \Gamma_i = \{\gamma^{(0)}\}$ where $(g, \gamma^{(0)}) = 1$ for all $g \in G$ (here (g, γ) denote the value of a character γ at a point g).

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For each $n \in \mathbb{Z}$ the group Γ_{-n} is the annulator of G_n, i.e.,

$$\Gamma_{-n} = G_n^{\perp} = \{\gamma \in \Gamma : (g, \gamma) = 1 \ \ \text{for all} \ \ g \in G_n\}.$$

In the compact case $G = G_0$ the group Γ of characters is discrete and it can be represented as a sum of increasing chain of finite subgroups

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Convergence of this series at a point g is the convergence of its partial sums

$$S_n(g) = \sum_{\gamma \in \Gamma_{-n}} a_\gamma \gamma(g)$$

.

Haar measures μ_G and μ_Γ on G and Γ, respectively, and normalized so that $\mu_G(G_0) = \mu_\Gamma(\Gamma_0) = 1$. Since μ_G is translation invariant,

$$\mu_G(G_n) = \mu_G(K_n) = \frac{1}{m_n}$$

for all the cosets K_n, $n \geq 0$.
We make those measures to be complete in a usual way.

Derivation basis $\mathcal{B}$ on the measure space $(G, \mathcal{M}, \mu_G)$.

Taking any function $\nu : G \to \mathbb{Z}$, we define a basis set by

$$\beta_\nu = \{(I, g) : g \in I,\ I = K_n(g),\ n \geq \nu(g)\}.$$

So our basis $\mathcal{B} = \mathcal{B}_G$ in G is the family $\{\beta_\nu : \nu \in \mathbb{Z}^G\}$.

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$\mathcal{B}$ has the filter base property: $\emptyset \notin \mathcal{B}$ and for every $\beta_1, \beta_2 \in \mathcal{B}$ there exists $\beta \in \mathcal{B}$ such that $\beta \subset \beta_1 \cap \beta_2$.

β-partition is a finite collection π of elements of β, where the distinct elements (I', x') and (I'', x'') in π have I' and I'' nonoverlapping, i.e., $\mu(I' \cap I'') = 0$.

Let $L \in \mathcal{I}$. If $\cup_{(I,x)\in\pi} I = L$ then π is called β-partition of L.

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Our basis $\mathcal{B}$ has partitioning property:

(i) for each finite collection $I_0, I_1, ..., I_n$ of $\mathcal{B}$-intervals with $I_1, ...I_n \subset I_0$ the difference $I_0 \setminus \cup_{i=1}^{n} I_i$ can be expressed as a finite union of pairwise non-overlapping $\mathcal{B}$-intervals;

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It follows easily from compactness of any $\mathcal{B}$-interval and the fact that any two $\mathcal{B}$-intervals K' and K'' are either disjoint or one of them is contained in the other one.

Let R be complexification of a Dedekind complete Riesz space.

Definition

We say that $f : L \to R$ is Henstock-Kurzweil integrable on a $\mathcal{B}$-interval L w.r. to $\mathcal{B}$ (in brief, *$H_{\mathcal{B}}$-integrable*) if there exists $A \in R$ such that

$$\inf_{\nu} \left(\sup \left\{ \left| \sum_{(I,g)\in\pi} f(g)\,\mu(I) - A \right| : \pi \text{ is a } \beta_\nu\text{-partition of } L \right\} \right) = 0.$$

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Theorem

f is $H_{\mathcal{B}}$-integrable on L $\Leftrightarrow$

$$\inf_{\nu}\left(\sup\left\{\sum_{(I,g)\in\pi} |f(g)\,\mu(I) - F(I)| : \pi \text{ is a } \beta_\nu\text{-partition of } L\right\}\right) = 0.$$

Let $\tau : \mathcal{I} \to R$ be a $\mathcal{B}$-interval function additive on $\mathcal{I}$

τ is said to be (o)-continuous at $g \in G$ w.r. to the basis $\mathcal{B}$ if

$$(o) \lim_{n\to\infty} \tau(K_n(g)) = 0.$$

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τ is (o)-differentiable at $g \in G$ with respect to $\mathcal{B}$ if

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exists. We denote it by $D^{(o)}_{\mathcal{B}}\tau(g)$.

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τ is (u)-differentiable on E with respect to the basis $\mathcal{B}$ if there exists a function $f : E \to R$ such that

$$\inf_{\nu} \left[\sup \left\{ \left| \frac{\tau(I)}{\mu(I)} - f(g) \right| : (I, g) \in \beta_\nu, \ g \in E \right\} \right] = 0.$$

f is called the (u)-derivative of $\tau$with respect to $\mathcal{B}$ on E and denoted by $D^{(u)}_{\mathcal{B}} \tau(g)$.

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To show that this (o)-limit is equal to zero, we define a monotone (o)-sequence

$$\varphi_n(s) = \begin{cases} m_j & \text{if} \quad s \in K_j(g) \setminus K_{j+1}(g) \quad j \geq n \\ 0 & \text{if} \quad s \in G \setminus K_n(g). \end{cases}$$

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By construction $m_n \chi_{K_n(g)}(s) \leq \varphi_n(s)$ and $\inf \varphi_n = 0$. Therefore $D^{(o)}_{\mathcal{B}} F(g) = 0$ for all g, while F is not trivial.

Theorems on recovering the primitives by the $H_{\mathcal{B}}$-integral

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Theorem 1

If a function $\tau : \mathcal{I} \to R$ is (u)-differentiable with respect to $\mathcal{B}$ on $\mathcal{B}$-interval L then derivative $D_{\mathcal{B}}^{(u)}\tau(g)$ is $H_{\mathcal{B}}$-integrable on L, and

$$(H_{\mathcal{B}}) \int_L D_{\mathcal{B}}^{(u)}\tau(g) = \tau(L).$$

A Riesz space satisfies property σ if, given any sequence $(u_n)_n$ in R with $u_n \geq 0 \; \forall \, n \in N$, there exists a sequence $(\lambda_n)_n$ of positive real numbers, such that the sequence $(\lambda_n u_n)_n$ is bounded in R.

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A Dedekind complete Riesz space is said to be regular if it satisfies property σ and if for each sequence $(r_n)_n$ in R, order convergent to zero, there exists a sequence $(l_n)_n$ of positive real numbers, with $\lim_n l_n = +\infty$, such that the sequence $(l_n r_n)_n$ is order convergent to zero.

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(Recall that in a regular Riesz space the (o)-convergence is equivalent to the (ru)-convergence.)

Theorem 2

Let R be a complexification of a regular Riesz space, $f : L \to R$ and $\tau : \mathcal{I} \to R$ such that for some countable set $Q \subset L$ the function f is the (u)-derivative $D^{(u)}_{\mathcal{B}} \tau(g)$ on $L \setminus Q$ and τ is (o)-continuous on Q with respect to $\mathcal{B}$. Then f is $H_{\mathcal{B}}$-integrable on L, and

$$(H_{\mathcal{B}}) \int_L f = \tau(L).$$

Application to series with respect to characters of G

For partial sums $S_n(g)$ of the series $\sum_{\gamma\in\Gamma} a_\gamma \gamma$ we consider pointwise (o)-convergence as well as (u)-convergence on a set.

Application to series with respect to characters of G

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Definition

Let E be any nonempty set, and $D = \mathbb{N}^E$. We say that the sequence of R-valued functions $(S_n(g))_n$, is (u)-convergent to the function $S(g)$ if there exists an (o)-net $(p_\nu)_{\nu \in D}$ such that $\forall\, \nu \in D$ we have:

$$\sup\{|S_n(g) - S(g)| : g \in E,\ n \geq \nu(g)\} \leq p_\nu.$$

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We associate with the series $\sum_{\gamma\in\Gamma} a_\gamma \gamma$ a set function F defined on each coset K_n by $F(K_n) = \int_{K_n} S_n(g) d\mu_G$. It can be checked that F is additive on $\mathcal{I}$.

Since the sum S_n is constant on each K_n, the definition of $F(K_n)$ implies

$$S_n(g) = \frac{F(K_n(g))}{\mu_G(K_n(g))}.$$

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As a consequence we get:

Lemma 1

If the partial sums $S_n(g)$ of the series $\sum_{\gamma\in\Gamma} a_\gamma \gamma$ are (o)-convergent to a function f at a point g then the associated function F is (o)-differentiable to f at g. If the partial sums $S_n(g)$ are (u)-convergent to a function f on a set E then the associated function F is (u)-differentiable to f on E.

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Lemma 2

Let R be a regular Riesz space and the coefficients $a_\gamma \in R$ of the series satisfy condition

$$(o) \lim_{n\to+\infty} sup\{|a_\gamma| : \gamma \in \Gamma_{-(n+1)} \setminus \Gamma_{-n}\} = 0.$$

Then the associated function F is (o)-continuous at each point g.

Theorem 3

The series $\sum_{\gamma\in\Gamma} a_\gamma \gamma$ is the $H_{\mathcal{B}}$-Fourier series of some $H_{\mathcal{B}}$-integrable function f if and only if the function F associated with this series coincides on each $\mathcal{B}$-interval I with the indefinite integral $(H_{\mathcal{B}}) \int_I f$.

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Theorem 4

If R is a regular Riesz space and a series $\sum_{\gamma\in\Gamma} a_\gamma \gamma$ is (u)-convergent to a function f on a set $G \setminus E$, where E is a countable subset of G, and the coefficients a_γ satisfy the above condition, then f is $H_\mathcal{B}$-integrable on G and the series is the Fourier series of f in the sense of the $H_\mathcal{B}$-integral.

Theorem 5 (for locally compact case)

If the the sequence of integrals

$$(H_{\mathcal{B}_\Gamma}) \int\limits_{\Gamma_{-n}} a(\gamma)(g,\gamma)d\mu_\Gamma$$

(u)-converges on G to $f(g)$, where $a(\gamma)$ is (o)-continuous on Γ, then f is $H_{\mathcal{B}_G}$-integrable on G_{-n} for each n and

$$a(\gamma) = (o) \lim_{n\to\infty} (H_{\mathcal{B}_G}) \int_{G_{-n}} f(g)\overline{(g,\gamma)}d\mu_G \quad \text{a.e. on } \Gamma. \tag{1}$$

THANK YOU FOR YOUR ATTENTION!!