An associated linear operator for a given nonlinear operator

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Introduction

Any Dedekind complete Banach lattice, E, with a quasi-interior point, e, is lattice isomorphic to a space of continuous, extended real-valued functions defined on a compact Hausdorff space X.

Conditions on the Operator

In addition to $T : E \to \mathbb{R}$ being nonlinear, T must also be:

- Orthogonally Additive
- Continuous
- Monotonic
- Subhomogeneous

Definitions

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An operator $T : E \to F$ between two Banach lattices is called **monotonic** if $T(x) \leq T(y)$ whenever $x \leq y$.

Definitions

Definition

The operator T is **subhomogeneous** if for $x \geq 0$ and $\alpha > 0$, there exist positive constants $m(\alpha)$ and $M(\alpha)$ with $m(\alpha)$ a monotone function of α and unbounded so that $m(\alpha)T(x) \leq T(\alpha x) \leq M(\alpha)T(x)$ and $M(\alpha)$ goes to zero as α goes to zero.

Example of the Types of Operators We Are Studying

The nonlinear operator $T : C(X) \to \mathbb{R}$ defined by $T(f) = (L(f))^2$ where L is a linear functional and $f \in C(X)$ is an example of an operator that satisfies all of the above conditions. More generally, other functions composed with $L(f)$, that is $T = \phi(L(f))$, would also satisfy these conditions.

Support of an Operator

The **support of** T, denoted K_T, is defined as
$K_T = X \setminus \bigcup_{\{h \in I(e)^+ : T(h)=0\}} \{x \in X : h(x) > 0\}$

Riesz Representation Theorem

Let X be a compact Hausdorff space and $C(X)$ the space of continuous real-valued functions on X. Then to each bounded linear functional F on $C(X)$ there corresponds a unique finite signed Baire measure μ on X such that $F(f) = \int f d\mu$ for each f in $C(X)$.

Motivation

In the nonlinear setting, we no longer have the concept of integration. In order to extend our results from the linear setting into the nonlinear setting, we needed to develop a complete measure related to the nonlinear operator T.

Setting

Let E be a Banach lattice with an order continuous norm and a quasi-interior point and let $T : E \to \mathbb{R}$ be as before. Notice in this setting, if we have a clopen set K, then the function χ_K is continuous.

First Definition of the Measure

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- $\mu^*(\mathcal{C}) = \wedge(\mu^*(\mathcal{A})) = \wedge(T(\chi_\mathcal{A}))$ for closed sets $\mathcal{C}$ and clopen sets $\mathcal{A}$ where $\mathcal{C}$ is contained in $\mathcal{A}$

Proposition

Proposition

Let $\{\mathcal{O}\}$ be the collection of open sets, let $\{\mathcal{A}\}$ be the collection of clopen sets such that $\mathcal{O} \subset \mathcal{A}$, and let $\{\mathcal{B}\}$ be the collection of clopen sets such that $\mathcal{B} \subset \mathcal{O}$. Then, $\vee T(\chi_{\mathcal{B}}) = \wedge T(\chi_{\mathcal{A}})$.

Proof Highlights

- E is Dedekind complete ($\vee\chi_{\mathcal{B}}$ and $\wedge\chi_{\mathcal{A}}$ exist)
- X is extremally disconnected and normal ($\vee\chi_{\mathcal{B}} = \wedge\chi_{\mathcal{A}}$)
- $\{\chi_{\mathcal{B}}\}$ is an upward directed set
- $\{\chi_{\mathcal{A}}\}$ is a downward directed set
- E has an order continuous norm and T is continuous

Second Definition of the Measure

Definition

For any open set $\mathcal{O}$, define $\mu^*(\mathcal{O}) = \vee T(\chi_{\mathcal{B}}) = \wedge T(\chi_{\mathcal{A}})$, where $\mathcal{B}$ and $\mathcal{A}$ are clopen sets such that $\mathcal{B} \subset \mathcal{O} \subset \mathcal{A}$.

Lemma

Lemma

$\mu^*(K) = T(\chi_K)$ for clopen sets K

Final Definition of the Measure

Definition

For any set $E \subset X$, define

$$\mu^*(E) = inf\{\mu^*(\mathcal{O}) : \mathcal{O} \text{ is open and } E \subset \mathcal{O}\}$$

Theorem

Theorem

μ^* is an outer measure

Outer Measure

To show μ^* is an outer measure, we must show that it satisfies the following three conditions:

1. $\mu^*(\emptyset) = 0$
2. $A \subset B \Rightarrow \mu^*(A) \leq \mu^*(B)$
3. $E \subset \bigcup_{i=1}^{\infty} E_i \Rightarrow \mu^*(E) \leq \sum_{i=1}^{\infty} \mu^*(E_i)$

Countable Subadditivity Proof Highlights

- Start with 2 open sets
- Construct disjoint clopen sets
- Use orthogonal additivity and monotonicity of T
- Repeat for a finite collection of open sets
- Repeat for a countably infinite collection of open sets
- Prove for any countably infinite collection of sets

Theorem

Theorem

μ^* is a topologically regular outer measure

Topologically Regular Outer Measure

To show μ^* is a topologically regular outer measure, we must show that it satisfies the following three conditions:

1. For each $E \subset X$, $\mu^*(E) = inf\{\mu^*(\mathcal{O}_i) : \mathcal{O}$ is open and $E \subset \mathcal{O}\}$
2. $\mu^*(\mathcal{O}_1 \cup \mathcal{O}_2) = \mu^*(\mathcal{O}_1) + \mu^*(\mathcal{O}_2)$ if $\mathcal{O}_1$ and $\mathcal{O}_2$ are disjoint open sets
3. $\mu^*(\mathcal{O}) = sup\{\mu^*(K) : K \subset \mathcal{O}$ for K compact$\}$ for $\mathcal{O}$ open.

Complete Measure

So, we have created a topologically regular outer measure μ^*. Now we can define a set E to be μ^*-measurable if for every set A, we have $\mu^*(A) = \mu^*(A \cap E) + \mu^*(A \cap \tilde{E})$. Let $\mathcal{M}$ be the collection of these measurable sets. Then $\mathcal{M}$ is a σ-algebra, and μ^* restricted to $\mathcal{M}$ is a complete measure on $\mathcal{M}$. We will call this measure μ.

Defining L

Using the complete measure μ we just found, define
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- L and T agree on the characteristic functions.
- The support of L is equal to the support of T.

Proof Highlights

- χ_K is continuous
- Consider elements not in the support of T
- T is monotonic, positive, and subhomogeneous
- L and T agree on characteristic functions
- $K_L \subset K_T$
- L is monotonic, positive, and linear
- $K_T \subset K_L$

Generalized Definition

Definition

We say L is **associated** with T if there exists a quasi-interior point, u of E such that for every decomposition $u = u_1 + u_2$ of u where $u_1 \wedge u_2 = 0$ and u_1 and u_2 are greater than or equal to zero, then $L(u_1) = T(u_1)$ and $L(u_2) = T(u_2)$.

Theorem

Theorem

If L is associated with T, then the support of L is equal to the support of T.

Proof Highlights

- u bounded
- u unbounded
- $u_n = ne \wedge u$
- $\alpha_n u_n \chi_H$
- Subhomogeneous or linear, monotonic, positive, and continuous

Conclusion

We have discussed properties of a linear operator associated with the nonlinear operator T and established the existence of an associated linear operator via the measure μ. We can now utilize the linear operator and the measure μ in the study of the nonlinear operator.

Questions

Thank you!

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