Non-commutative analysis — modern advances

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Noncommutative analysis

Noncommutative analysis is an analysis of functions whose arguments do not commute and whose values do not commute. We will discuss

- differentiation,
- Taylor-type approximation,
- applications to mathematical physics /perturbation theory.

Functions are defined on operators acting in a separable Hilbert space (can think of finite matrices). In the first part of the talk, H and V are self-adjoint. H is an initial operator and V its perturbation.
Given a Borel function $f : \mathbb{R} \mapsto \mathbb{C}$ bounded on the spectrum on H, the operator function $f(H)$ is defined via the spectral theorem (functional calculus).

Spectral theorem
(diagonalisation of an operator)

- Finite-dimensional case: $H = H^*$ is a finite matrix with eigenvalues $\{\lambda_k\}_{k=1}^n$. If P_k is a projection onto the eigenspace corresponding to λ_k, then

$$H = \sum_{k=1}^{n} \lambda_k \cdot P_k.$$

- Infinite-dimensional case: $H = H^*$ is an operator. There is a unique spectral measure E such that

$$H = \int_{\mathbb{R}} \lambda \, dE(\lambda) = \int_{\mathrm{spectrum}(H)} \lambda \, dE(\lambda).$$

For Borel $f : \mathbb{R} \mapsto \mathbb{C}$ bounded on $\sigma(H)$, the function of H is defined by

$$f(H) := \int_{\mathbb{R}} f(\lambda) \, dE(\lambda).$$

I.M. Lifshits, *Physics Literature* '47 — *Uspehi Mat. Nauk* '52

If H describes interactions between the atoms of a pure crystal and $H+V$ of a crystal with impurities, then the change in the free energy of a crystal equals (here, tr is the standard trace)

$$\mathrm{tr}\big[f(H+V)-f(H)\big],$$

for a small defect V (mathematically, if the trace above is well defined). Lifshits was looking for efficient formulas to compute the change in the free energy.

Non-commutative Lipschitz estimates

- In 1953 M. G. Krein proved that if H and V are self-adjoint and $V \in \mathfrak{S}^1$, then $f(H+V) - f(H) \in \mathfrak{S}^1$ for every $f \in C_c^\infty$. Here, $\mathfrak{S}^p$ is the Schatten-von Neumann class.
- M.G. Krein asked (in the case $p = 1$): Let $V \in \mathfrak{S}^p$, $1 \le p \le \infty$ and if $f \in C^1(\mathbb{R})$. Is it true that

$$f(H+V) - f(H) \in \mathfrak{S}^p?$$

- A positive answer was relatively simply found for $p = 2$.
- Yu. Farforovskaya (and later others, e.g. Davies, Peller) showed that the answer is negative for $p = 1$ and $p = \infty$ in 1972 and 1967, respectively.

Theorem

The answer is positive if $1 < p < \infty$ (D. Potapov & F.S., Acta Math. 2011).

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Perturbation argument, in its simplest form

Let $A = \sum_j \lambda_j E_j$ and $B = \sum_k \mu_k F_k$. We argue as follows

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f(B) - f(A) &= \\
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Thus, we obtained

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f(B) - f(A) = T_{\psi_f}(A - B),
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T_\phi(X) = \sum_{j,k} \phi(\lambda_j, \mu_k) F_k X E_j
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Schur multipliers and Fourier multipliers

As we have seen on the previous frame, the analysis of the difference

$$f(B) - f(A)$$

can be reduced to the question about the behavior of the Schur multiplier

$$T_{\psi_f}, \text{ where } \psi_f(x,y) = \frac{f(x)-f(y)}{x-y}$$

on the element $A - B \in \mathfrak{S}^p$, $1 \leq p < \infty$. The study of various classes of Schur multipliers on Schatten-von Neumann classes $\mathfrak{S}^p$ is one of the active areas of Noncommutative Analysis. This study is a noncommutative counterpart of the classical Fourier analysis. We shall exploit this connection for the case when $1 < p \neq 2 < \infty$.

A $\mathfrak{S}^2$ estimate is simple

The following lemma is well known:

Lemma (non-commutative Parseval's identity)

If $X \in \mathfrak{S}^2$, then

$$\|X\|_2^2 = \sum_{j,k} \|F_k X E_j\|_2^2 ,$$

where $\{E_j\}$ and $\{F_k\}$ are families of orthogonal projections.

- This lemma ensures that T_{ψ_f} is bounded on $\mathfrak{S}^2$ as long as

$$\psi_f \in L^\infty \iff f' \in L^\infty .$$

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Vector-valued Harmonic analysis in UMD-spaces

- The new approach to $\mathfrak{S}^p$ have become possible due to recent (and not so recent) developments in the vector-valued Harmonic analysis.
- The key concept in this area is the concept of UMD (unconditional martingale differences) spaces introduced by Pisier and developed by Burkholder.
- One of the key results is the vector-valued Marcinkiewicz multiplier theorem due to J. Bourgain

Theorem

If X is a UMD Banach space, then the Fourier multiplier defined by

$$(\widehat{T_m(f)})(k) = m_k \hat{f}(k), \quad k \in \mathbb{Z}$$

is bounded on vector-valued Bochner space $L^p(\mathbb{T}, X)$ if m is a bounded sequence and m is of bounded variation over every dyadic interval $2^d \leq |k| < 2^{d+1}$ uniformly for $d \in \mathbb{N}$.

The new approach to $\mathfrak{S}^p$ with $1 < p < \infty$, $p \neq 2$

Unlike the simple case $p = 2$, the approach of D. Potapov & F.S. [Acta Math., 2011] is based on vector valued Marcinkiewicz multiplier theorem and the following ideas:

Lemma

There is a rapidly decreasing function h such that, for every $\|f'\|_\infty \leq 1$,

$$\psi_f(x,y) = \frac{f(x) - f(y)}{x - y} = \int_{\mathbb{R}} h(\sigma)\, |f(x) - f(y)|^{i\sigma}\, |x - y|^{-i\sigma}\, d\sigma.$$

- The operator $R_\sigma = T_{w_\sigma}$, where $w_\sigma(x,y) = |x - y|^{i\sigma}$ is linked with the Calderon-Zygmund theory of vector-valued singular integral operators, in particular, with the Marcinkiewicz multiplier theorem.
- The representation above allows to write our Schur multiplier as follows

$$T_{\psi_f} = \int_{\mathbb{R}} h(\sigma)\, \tilde{R}_\sigma \cdot R_{-\sigma}\, d\sigma.$$

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The new approach to $\mathfrak{S}^p$ with $1 < p < \infty$, $p \neq 2$

computing the total variation of the sequence $\lambda = \{n^{is}\}_{n>0}$ over dyadic intervals via the fundamental theorem of the calculus, we have

$$\left|n^{is} - (n+1)^{is}\right| \leq \frac{|s|}{n}, \quad n \geq 1$$

and thus immediately

$$\sum_{2^k \leq n \leq 2^{k+1}} \left|n^{is} - (n+1)^{is}\right| \leq |s|, \quad k \geq 0.$$

Together with the vector valued Marcinkiewicz multiplier theorem and Transference Method (developed, in particular, by Berkson and Gillespie), we infer that $\|R_{-\sigma}\|_{\mathfrak{S}^p \to \mathfrak{S}^p} \leq (1 + |s|)$. A similar estimate also holds for $\tilde{R}_\sigma$. This allows us to conclude that $\|T_{\psi_f}\|_{\mathfrak{S}^p \to \mathfrak{S}^p} < \infty$. We are done.

Spectral shift function of M.G. Krein

Answering Lifshits's question, computing

$$\operatorname{tr}\left(f(H+V)-f(H)\right),$$

M.G. Krein introduced an object known now as a spectral shift function of Krein (the function ξ below).

Theorem (M.G. Krein, 1953)

If H and V are self-adjoint and $V \in \mathfrak{S}^1$, then there is L^1-function $\xi = \xi_{H,V}$ such that

$$\operatorname{tr}\left(f(H+V)-f(H)\right) = \int_{\mathbb{R}} f'(t)\,\xi(t)\,dt,$$

for every $f \in C_c^\infty$.

Construction of the Krein's function in case of finite trace

Let $H, V \geq 0$, let τ be finite trace and let $n_H(t) := \tau(E_H(t, \infty)), \quad t \in \mathbb{R}$.
It follows from the functional calculus that $f(H) = -\int_0^\infty f(s) dE_H(s, \infty)$.
Taking the trace and integrating by parts, we obtain

$$\begin{aligned}\tau(f(H)) &= -\int_0^\infty f(s) dn_H(s) \\ &= -f(s)n_H(s)\Big|_0^\infty + \int_0^\infty f'(s)n_H(s)ds = \int_0^\infty f'(s)n_H(s)ds.\end{aligned}$$

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Let $H, V \geq 0$, let τ be finite trace and let $n_H(t) := \tau(E_H(t,\infty)), \quad t \in \mathbb{R}$.
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Spectral shift function explained

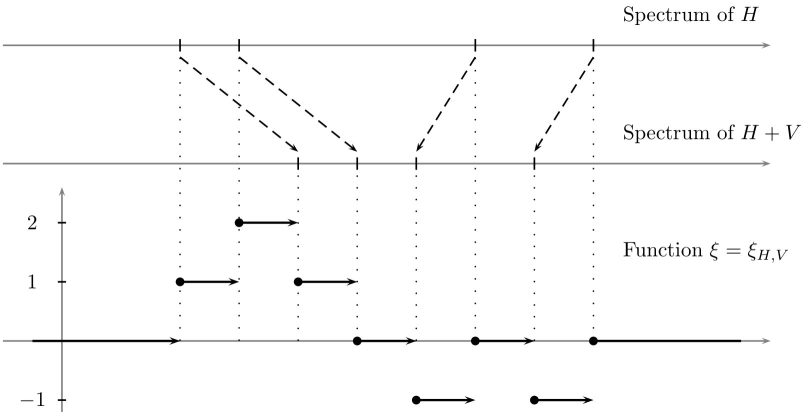

Figure: M.G. Krein spectral shift function explained

L. Koplienko development of the trace formula

In 1984, L. Koplienko proved the following improvement of the M.G. Kreins trace formula:

Theorem (Koplienko, 1984)

If H and V are self-adjoint and $V \in \mathfrak{S}^2$, then there is an L^1-function η (the spectral shift function of Koplienko) such that

$$\mathrm{tr}\,(R_2(f,H,V)) = \int_{\mathbb{R}} f''(t)\,\eta(t)\,dt, \quad \text{for every } f \in C_c^\infty,$$

where $R_2(f,H,V) = f(H+V) - f(H) - \frac{d}{dt}\left[f(H_t)\right]\Big|_{t=0}$ *and* $H_t := H + tV$.

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- A complex combination of vector-valued harmonic analysis enabled the following result (which lies at the core of our resolution of Koplienko conjecture of 1984):

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Let H and V be self-adjoint operators and let $1 < p < \infty$, $n > 1$. There is $c_{n,p} > 0$ such that,

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