Riesz Space Valued Integral

Neşet Özkan TAN
Delft University of Technology

Positivity VII

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Contents

Let $\mathcal{F}$ be an algebra of subsets of X and E Dedekind complete Riesz space. Then the followings are hold:
(*) Let $\mu_1, \mu_2 \in oba(\mathcal{F}, E)$ and λ, τ define from $\mathcal{F}$ to E by the following:

$$\lambda(A) = \sup\{\mu_1(B) + \mu_2(A \setminus B), B \subset A, B \in \mathcal{F}\}$$

and

$$\tau(A) = \inf\{\mu_1(B) + \mu_2(A \setminus B), B \subset A, B \in \mathcal{F}\}.$$

Then $\lambda, \tau \in oba(\mathcal{F}, E)$.
(**) $oba(\mathcal{F}, E)$ is Dedekind complete Riesz Space.

Equivalences of Order countably additivity

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EQUIVALENCES OF ORDER COUNTABLY ADDITIVITY

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(4) For each decreasing sequence A_n in $\mathcal{F}$ such that $\bigcap_{n\geq 1} A_n = \emptyset$,

$$o - \lim \mu(A_n) = 0.$$

Approach with cardinality of $\mathcal{F}$

Let $\mathcal{F}$ be an algebra of subsets of X and E Dedekind complete Riesz space. Then the followings are equal:
(1)Each additive measure $\mu : \mathcal{F} \to E$ is order bounded.

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Let $\mathcal{F}$ be an algebra of subsets of X and E Dedekind complete Riesz space. Then the followings are equal:
(1)Each additive measure $\mu : \mathcal{F} \to E$ is order bounded.
(2)$\mathcal{F}$ is finite.

S-boundedness

Let μ be a measure from $\mathcal{F}$ to Dedekind complete Riesz space E. Where $\mathcal{F}$ is an algebra of subsets of nonempty set X. Then μ is called **s-bounded** if for each pairwise disjoint sequence (A_n) $o - \lim \mu(A_n) = 0$ is hold.

PR Property

Let E be a Riesz space and (h_k) be a increasing sequence of positive elements in E. If for each $e \in E$ there exists $t > 0$ and k such that $|e| \leq th_k$ then E has **PR property**.

A result from Schwartz, C.

Let $\mathcal{F}$ be an algebra of subset of X and E be a Riesz space has PR property. Then the followings are equivalent:
(1) $\mu : \mathcal{F} \to E$ is order bounded.
(2) $\mu : \mathcal{F} \to E$ is s-bounded.

The Outer Measure Generated by a Measure

For Dedekind complete Riesz space valued, positive measure μ, the function $\mu^* : \wp(X) \to [0, \infty)$ is defined by,

$$\mu^*(A) = \inf\{\tilde{e}(\mu(B)) : A \subset B, B \in \mathcal{F}\}.$$

is a real valued outer measure for each strictly positive order continuous functional.

Prop. of Outer Measure

Let μ_1 and μ_2 be in $oba(\mathcal{F}, E)$ positive measures. Then we have,

$$(\mu_1 + \mu_2)^* = \mu_1^* + \mu_2^*.$$

Particularly,
For all $\mu \in oba(\mathcal{F}, E)$, we have $|\mu|^* = \mu_+^* + \mu_-^*$.

NULL SET

A subset A of X is called μ**-null set** if $|\mu|^*(A) = 0$.

Prop. of Null sets

Let $(X, \mathcal{F}, \mu)$ be a measure space. Then the following are hold.
(1) $\emptyset$ is a μ-null set.
(2) $B \subset A$ is a μ-null set whenever A is a μ-null set.
(3) Finite union of μ-null set is a μ-null set.

Null Function

Let $\varepsilon > 0$ be given. f is called μ-**null function** if

$$|\mu|^*(\{x \in X : \tilde{e}(|f(x)|) > \varepsilon\}) = 0.$$

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Two functions f and g are called **almost everywhere equal** if $f - g$ is μ-null function. In this case we write "**f=g** μ**-a.e**"
f is called, **dominated almost everywhere by** g if $f \leq g + h$ for a μ-null function h.

Prop. of Null Functions

(1) Let f, g be μ-null functions. Then $cf + dg$ and $|f|$ are μ-null functions for real numbers c, d.
(2) Let f be a μ-null function. If $|g| \leq |f|$ then g is a μ-null function.

Hazily Convergence

Let f_n be a sequence of functions define on X to E. f_n is called convergence to function f hazily if for all $\varepsilon > 0$,

$$\lim_{n\to\infty} |\mu|^*(\{x \in X : \tilde{e}(|f_n(x) - f(x)|) > \varepsilon\}) = 0.$$

UNIQUENESS OF HAZILY CONVERGENCE

Let f_n be a sequence of functions such that hazily convergence to f and g. Then $f = g$ μ-a.e.. Furthermore if f_n is a sequence functions such that hazily converge to f and $f = g$ μ-a.e. then f_n hazily convergence to g.

Prop. of Hazily conv.

Let (f_n) and (g_n) be sequence from X to E such that hazily convergent to f and g, respectively.Then the followings are hold.
(1) Let c and d be two real numbers than sequence of $(cf_n + dg_n)$ hazily converge to $cf + dg$.
(2) The sequence of $(f_n \vee g_n)$ hazily converge to $f \vee g$.

Hazily convergence and Order convergence

Let (f_n) be sequence of functions such that order converge to $f : X \to E$ in E^X then (f_n) is hazily converge to f.

Simple Function

$\varphi : X \to E$ is called E**-valued simple function** if φ is finite valued $x_1, x_2, ..., x_n \in E$ and $\varphi^{-1}(\{x_i\}) = A_i \in \mathcal{F}$ for each $i = 1, .., n$. Such as functions φ, following formula is called **standard representation** of φ.

$$\varphi(x) = \sum_{i=1}^{n} \chi_{A_i}.xi$$

Int. of Simple Functions

Integral of E valued simple function which have standard representation such as before, is defined by following finite sum:

$$\int \varphi d(\tilde{e}, |\mu|) = \sum_{i=1}^{n} \tilde{e} \otimes |\mu|(A_i).x_i$$

For the integral vector we will use the notation $\int \varphi d(\tilde{e}, |\mu|)$. Where $\tilde{e} \otimes |\mu|(A_i)$ is $\tilde{e}(|\mu|(A_i))$.

Int. is independent from represent.

If the following formula is different representation of φ

$$\varphi(x) = \sum_{j=1}^{m} \chi_{B_j}.y_j,$$

for each $j = 1, .., m$ and $y_j \neq 0$. Then we have,

$$\int \varphi d(\tilde{e}, |\mu|) = \sum_{j=1}^{m} \tilde{e} \otimes |\mu|(B_j).y_j.$$

μ-A.E. AND INT.

Let φ, and ϕ be E-valued μ-a.e. simple function. Then

$$\int \varphi d(\tilde{e}, |\mu|) = \int \phi d(\tilde{e}, |\mu|)$$

PROP. OF INT. OF SIMP. FUNC.

(1) Let φ be E-valued simple function, for all $A \in \mathcal{F}$,

$$\int_A \phi d(\tilde{e}, |\mu|) = \int_A \phi d(\tilde{e}, \mu_+) + \int_A \phi d(\tilde{e}, \mu_-)$$

and

$$|\int_A \phi d(\tilde{e}, |\mu|)| \leq \int_A |\phi| d(\tilde{e}, |\mu|).$$

(2)Let φ be E-valued simple function on X and if $\varphi \geq 0$ μ-a.e. then for each $A \in \mathcal{F}$,

$$\int_A \varphi d(\tilde{e}, |\mu|) \geq 0.$$

Prop. of Simple Func.

Let φ be E-valued simple function on X and if $\varphi \geq \phi$ μ-a.e. then for each $A \in \mathcal{F}$,

$$\int_A \varphi d(\tilde{e}, |\mu|) \geq \int_A \phi d(\tilde{e}, |\mu|).$$

Prop. of Simple Func.

(3) Let φ and ϕ be E-valued simple functions. For each $A \in \mathcal{F}$ following inequities are hold.

$$\begin{aligned} \left| \int_A |\varphi| d(\tilde{e}, |\mu|) - \int_A |\phi| d(\tilde{e}, |\mu|) \right| &\leq \int_A ||\varphi| - |\phi|| d(\tilde{e}, |\mu|) \\ &\leq \int_A |\varphi + \phi| d(\tilde{e}, |\mu|) \\ &\leq \int_A |\varphi| d(\tilde{e}, |\mu|) + \int_A |\phi| d(\tilde{e}, |\mu|) \end{aligned}$$

Absolute Continuity

Let $\mu \in oba(\mathcal{F}, E)$ and ν be a measure from $\mathcal{F}$ to E. Also let $\tilde{e}(|\nu(A)|) < \varepsilon$ is satisfied for each $A \in \mathcal{F}$.Then ν is called **absolute continuous with respect to** μ if there exist $\delta > 0$ depends ε such that $\tilde{e}(|\mu|(A)) < \delta$.

ABSOLUTE CONTINUITY OF SIMPLE FUNC.

If φ is E-valued simple function then ν is defined by following formula

$$\nu(A) = \int_A \phi d(\tilde{e}, |\mu|)$$

for each $A \in \mathcal{F}$ is order bounded and absolute continuous measure with respect to μ.

μ ,$\tilde{e}$ AND INTEGRATION OF SIMPLE FUNCT.

Let $f : X \to E$ be a simple function then the following are hold.
(1)For order bounded measure $\nu : \mathcal{F} \to E$ if $|\nu| \leq |\mu|$ then for each $A \in \mathcal{F}$,

$$\int_A fd(\tilde{e}, |\nu|) \leq \int_A fd(\tilde{e}, |\mu|).$$

(2) Let $\tilde{l}$ be taken from order continuous dual of E. if $\tilde{l} \leq \tilde{e}$ then for each $A \in \mathcal{F}$,

$$\int_A fd(\tilde{l}, |\mu|) \leq \int_A fd(\tilde{e}, |\mu|)$$

Monotone conv. th. for Simple func.

Let sequence of simple functions $p_n : X \to E$ increase or decrease to zero in E^X. Then the sequence $\int p_n d(\tilde{e}, |\mu|)$ is increase or decrease to zero of E, respectively. i.e.,

$$p_n \downarrow\uparrow 0 \Rightarrow \int p_n d(\tilde{e}, |\mu|) \downarrow\uparrow 0$$

Monotone convergence theorem for simple functions

Let increasing or decreasing sequence of simple functions $f_n : X \to E$ order converge to function $f : X \to E$ in E^X. Then the sequence $\int p_n d(\tilde{e}, |\mu|)$ is converge to $\int f d(\tilde{e}, |\mu|)$ in E.

Main theorem for integration

Let (φ_n) and (ϕ_n) be sequences of simple functions which are converge to f hazily. Moreover assume that following equality hold.

$$o-\lim_{m,n\to\infty}\int|\phi_n-\phi_m|d(\tilde{e},|\mu|)=o-\lim_{m,n\to\infty}\int|\varphi_n-\varphi_m|d(\tilde{e},|\mu|)=0$$

Then followings limits are exists and for each $A\in\mathcal{F}$, we have

$$o-\lim_{n\to\infty}\int\phi_n d(\tilde{e},|\mu|)=o-\lim_{n\to\infty}\int\varphi_n d(\tilde{e},|\mu|).$$

Def. of integrable func.

Let $(X, \mathcal{F}, \mu, \tilde{e})$ be a measure space. An E-valued function f is called integrable with respect to μ if:
(1) There exists a sequence of functions (f_n) such that hazily convergence to f.

Def. of integrable func.

Let $(X, \mathcal{F}, \mu, \tilde{e})$ be a measure space. An E-valued function f is called integrable with respect to μ if:
(1) There exists a sequence of functions (f_n) such that hazily convergence to f.
(2) And $o - \lim_{m,n\to\infty} \int |f_n - f_m| d(\tilde{e}, |\mu|) = 0$.

μ AND $\tilde{e}$

Let $f : X \to E$ be an integrable function then the followings are hold.
(1) If $\nu : \mathcal{F} \to E$ is a order bounden measure such that s$|\nu| \leq |\mu|$ then for each $A \in \mathcal{F}$,

$$\int_A fd(\tilde{e}, |\nu|) \leq \int_A fd(\tilde{e}, |\mu|).$$

(2) If $\tilde{l}$ is taken from order continuous dual of E such that $\tilde{l} \leq \tilde{e}$ then,

$$\int_A fd(\tilde{l}, |\mu|) \leq \int_A fd(\tilde{e}, |\mu|).$$

APPROACH WITH SIMPLE FUNCTIONS

Let f be an integrable function and (f_n) be determining sequence of f. In this case $f_n - f$ is integrable for each $n \in \mathbb{N}$, Moreover,

$$o - \lim_{n \to \infty} \int |f_n - f| d(\tilde{e}, |\mu|) = 0.$$

PROP. OF INTEGRAL

Let $(X, \mathcal{F}, \mu, \tilde{e})$ be a measure space.
(1) If f is E-valued integrable function with respect to μ. Then f is integrable with respect to μ_+ and μ_-. Furthermore for each $A \in \mathcal{F}$ we have,

$$\int_A f d(\tilde{e}, |\mu|) = \int_A f d(\tilde{e}, \mu_+) + \int_A f d(\tilde{e}, \mu_-)$$

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(2) If f, g are E-valued integrable functions with respect to μ. For real numbers $c, d \in \mathbb{R}$ E-valued functions $cf + dg$ and $|f|, f_+$, $f_-, f \vee g, f \wedge g$ are also integrable with respect to μ. Particularly for each $A \in \mathcal{F}$ we have,

$$|\int_A fd(\tilde{e}, |\mu|)| \leq \int_A |f| d(\tilde{e}, |\mu|)$$

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Prop. of İnt.

and

$$\int_A f d(\tilde{e}, |\mu|) + \int_A g d(\tilde{e}, |\mu|) = \int_A f \vee g d(\tilde{e}, |\mu|) + \int_A f \wedge g d(\tilde{e}, |\mu|)$$

and

$$\int_A f d(\tilde{e}, |\mu|) = \int_A f_+ d(\tilde{e}, |\mu|) - \int_A f_- d(\tilde{e}, |\mu|).$$

(3) Let f be an integrable with respect to μ functions such that $f \geq 0$ μ-a.e. then for each $A \in \mathcal{F}$ we have

$$\int_A f d(\tilde{e}, |\mu|) \geq 0.$$

Prop of int.

Particularly if g is integrable function with respect to μ such that $f \geq g$ μ-a.e., then for each $A \in \mathcal{F}$, we have

$$\int_A f d(\tilde{e}, |\mu|) \geq \int_A g d(\tilde{e}, |\mu|).$$

(4) Let f and g be integrable functions with respect to μ.Then for each $A \in \mathcal{F}$ we have,

$$\begin{aligned} \left| \int_A |f| d(\tilde{e}, |\mu|) - \int_A |g| d(\tilde{e}, |\mu|) \right| &\leq \int_A ||f| - |g|| d(\tilde{e}, |\mu|) \\ &\leq \int_A |f + g| d(\tilde{e}, |\mu|) \\ &\leq \int_A |f| d(\tilde{e}, |\mu|) + \int_A |g| d(\tilde{e}, |\mu|) \end{aligned}$$

ABSOLUTE CONTINUITY AND ORDER BOUNDEDNESS

Let f be E-valued integrable functions with respect to μ then,

$$\nu(A) = \int_A f d(\tilde{e}, |\mu|)$$

function for each $A \in \mathcal{F}$, is order bounded and absolute continuous with respect to μ.

LAST RESULT

Let f be E-valued integrable function with respect to μ. f is null function if and only if f is integrable with respect to μ and $\int |f| d(\tilde{e}, |\mu|) = 0$.. Furthermore if f is integrable with respect to μ and $f = g$ μ-a.e. then g is integrable with respect to μ. Particularly for each $A \in \mathcal{F}$ we have,

$$\int_A f d(\tilde{e}, |\mu|) = \int_A g d(\tilde{e}, |\mu|).$$

Thank you for your attention.