Random unconditionally convergent bases in Banach spaces

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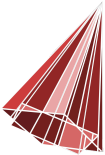
Joint work with J. López-Abad (CSIC)

Positivity VII



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Outline

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A basis $(x_n)_{n\in\mathbb{N}}$ of a Banach space X is unconditional provided for every $x \in X$ its expansion $\sum_{n\in\mathbb{N}} a_n x_n$ converges unconditionally.

Theorem

TFAE:

- *(x_n) is an unconditional basis.*
- *For every $A \subset \mathbb{N}$,*

$$\sum_{n\in\mathbb{N}} a_n x_n \text{ converges } \Rightarrow \sum_{n\in A} a_n x_n \text{ converges.}$$

- *For every choice of signs $(\theta_n)_{n\in\mathbb{N}}$,*

$$\sum_{n\in\mathbb{N}} a_n x_n \text{ converges } \Rightarrow \sum_{n\in\mathbb{N}} \theta_n a_n x_n \text{ converges.}$$

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Given $\theta = (\theta_n)_{n\in\mathbb{N}}$, let $M_\theta : X \to X$ given by

$$M_\theta\Big(\sum_{n\in\mathbb{N}} a_n x_n\Big) = \sum_{n\in\mathbb{N}} \theta_n a_n x_n$$

- (x_n) unconditional $\Leftrightarrow \sup_\theta \|M_\theta\| < \infty$.

Definition

A basis $(x_n)_{n\in\mathbb{N}}$ is **sub-unconditional** if there is $K \geq 1$ such that

$$\Big\|\sum_{n=1}^m a_n x_n\Big\| \leq K\mathbb{E}_\theta\Big(\Big\|\sum_{n=1}^m \theta_n a_n x_n\Big\|\Big) = K\frac{1}{2^m}\sum_{(\theta_n)\in\{-1,+1\}^m}\Big\|\sum_{n=1}^m \theta_n a_n x_n\Big\|.$$

Notice:

- (x_n) unconditional $\Leftrightarrow \|\sum_{n=1}^m a_n x_n\| \approx \mathbb{E}_\theta(\|\sum_{n=1}^m \theta_n a_n x_n\|)$.
- (x_n) sub-unconditional with $K = 1 \Rightarrow (x_n)$ unconditional.

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Questions

- What properties of an unconditional basis work for sub-unconditional?
- Does every sub-unconditional basis have an unconditional subsequence (resp. blocks)?
- Is every block of a sub-unconditional basis, also sub-unconditional?
- Can reflexivity be characterized somehow? (in the spirit of James theorem)
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Example

The summing basis (s_n) in c_0 does not have any sub-unconditional subsequence

$$\Big\|\sum_{n=1}^{m} a_n s_n\Big\| = \sup_{1\leq n\leq m}\Big|\sum_{j=1}^{n} a_j\Big|.$$

In particular,

$$\mathbb{E}_\theta\Big(\Big\|\sum_{n=1}^{m}\theta_n a_n s_n\Big\|\Big) = \int_0^1 \sup_{1\leq n\leq m}\Big|\sum_{j=1}^{n} a_j r_j(t)\Big| dt \approx \Big(\sum_{j=1}^{m} a_j^2\Big)^{\frac{1}{2}},$$

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Example

The unit basis (u_n) of James space is sub-unconditional

$$\|\sum_{n\in\mathbb{N}} a_n u_n\|_J = \sup\Big\{\big(\sum_{k=1}^{m}(a_{p_k} - a_{p_{k+1}})^2\big)^{\frac{1}{2}} : p_1 < p_2 < \ldots < p_{m+1}\Big\}.$$

It holds that

$$\|\sum_{i=1}^{m} a_i u_i\|_J \leq \sqrt{2}\,\mathbb{E}\Big(\Big\|\sum_{i=1}^{m}\theta_i a_i u_i\Big\|_J\Big).$$

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Every weakly null sequence (x_n) in $L^1[0,1]$ has a sub-unconditional subsequence.

Proof: [later]

Recall, $L^1[0,1]$ has no unconditional basis. Actually, $L^1[0,1]$ does not embed in a space with unconditional basis [Pelczynski (1961)].

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Every weakly null sequence (x_n) in $L^1[0,1]$ has a sub-unconditional subsequence.

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Definition

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$$\mathbb{E}_\theta\Big(\Big\|\sum_{n=1}^{m}\theta_n a_n x_n\Big\|\Big) \leq K\Big\|\sum_{n=1}^{m} a_n x_n\Big\|.$$

These bases carry also a different name: Random unconditionally convergent [Billard-Kwapien-Pelczynski-Samuel, Texas Funct. Anal. Seminar 1985]

- (x_n) unconditional $\Leftrightarrow$ (x_n) sub- and super-unconditional.
- (x_n) super-unconditional with $K = 1$ $\Rightarrow$ (x_n) unconditional.
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Proposition

Let (x_n) be a basis, and (x_n^*) its bi-orthogonal functionals.

- *(x_n) super-unconditional $\Rightarrow$ (x_n^*) sub-unconditional.*
- *(x_n^*) super-unconditional $\Rightarrow$ (x_n^*) sub-unconditional.*

However, if we take (s_n), the summing basis of c_0

$$s_n = (\overbrace{1, \ldots, 1}^{(n)}, 0 \ldots),$$

this is not super-uncondtional, although

$$s_n^* = (\overbrace{0, \ldots, 0}^{(n-1)}, 1 - 1, 0, \ldots)$$

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Recall James theorem: A Banach space X with unconditional basis which does not contain ℓ_1 nor c_0 subspaces, is reflexive.

Theorem

1. *Let (x_n) be a basis of a Banach space X such that every block is sub-unconditional. (x_n) is shrinking $\Leftrightarrow$ $\ell_1 \not\subset X$*
2. *Let (x_n) be a basis of a Banach space X such that every block is super-unconditional. (x_n) is boundedly complete $\Leftrightarrow$ $c_0 \not\subset X$.*

Does every weakly null sequence have a sub-unconditional sub-sequence?

NO [e.g. Maurey-Rosenthal space (Studia 1977).]

Is every block sequence of a sub-unconditional sequence also sub-unconditional?

NO [A modification of M-R.]

Given a sub-unconditional sequence, does it have an unconditional subsequence?

NO [e.g. a weakly null sequence in $L_1[0,1]$ without unconditional subsequences (Johnson-Maurey-Schechtman, JAMS 2007)]

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Theorem

X separable r.i. space on $[0,1]$ with finite upper Boyd index. Every weakly null sequence (x_n) in X has a sub-unconditional subsequence.

Proof:

(h_j) Haar basis on $[0,1]$: i.e. for $j = 2^k + l$, with $k \in \mathbb{N}$ and $1 \leq l \leq 2^k$,

$$h_j = \chi_{[\frac{2l-2}{2^{k+1}}, \frac{2l-1}{2^{k+1}})} - \chi_{[\frac{2l-1}{2^{k+1}}, \frac{2l}{2^{k+1}})}.$$

wlog $\exists p_k \leq q_k < p_{k+1}$

$$x_{n_k} = \sum_{j=p_k}^{q_k} b_j h_j$$

Given $(a_k)_{k=1}^m$, we can define the following martingale

$$f_n = \begin{cases} \sum_{k=1}^n a_k x_{n_k} & n < m \\ \sum_{k=1}^m a_k x_{n_k} & n \geq m. \end{cases}$$

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Using Johnson-Schechtman's inequality for Martingale differences

$$\Big\|\sum_{k=1}^m a_k x_{n_k}\Big\| \leq \|\sup_n |f_n|\| \lesssim \Big\|\Big(\sum_{k=1}^\infty |f_k - f_{k-1}|^2\Big)^{\frac{1}{2}}\Big\| = \Big\|\Big(\sum_{k=1}^m |a_k x_{n_k}|^2\Big)^{\frac{1}{2}}\Big\|.$$

Now, by Maurey-Khintchine (X has finite cotype), it holds that

$$\Big\|\Big(\sum_{k=1}^m |a_k x_{n_k}|^2\Big)^{\frac{1}{2}}\Big\| \lesssim \int_0^1 \Big\|\sum_{k=1}^m r_k(s) a_k x_{n_k}\Big\| ds.$$

Thus,

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Thank you for your attention.