The principal inverse of the gamma function

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The gamma function $\Gamma(x)$ is usually defined by the Euler form

$$\Gamma(x) = \int_0^\infty e^{-t} t^{x-1} dt$$

for $x > 0$. This is extended to $\Re z > 0$. By $\Gamma(z+1) = z\Gamma(z)$ for $z \neq 0, -1, -2, \cdots$ it is defined and holomorphic on $\mathbf{C} \setminus \{0, -1, -2, \cdots\}$

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The Weierstrass form

$$\frac{1}{\Gamma(x)} = xe^{\gamma x} \prod_{n=1}^{\infty} (1 + \frac{x}{n}) e^{-\frac{x}{n}} \tag{1}$$

is useful, where γ is the Euler constant defined by

$$\gamma = \lim_{n\to\infty} (1 + \frac{1}{2} + \cdots + \frac{1}{n} - \log n) = 0.57721 \cdots .$$

From (1) (Weierstrass form) it follows that

$$\log\Gamma(x) = -\log x - \gamma x + \sum_{n=1}^{\infty}\left(\frac{x}{n} - \log(1+\frac{x}{n})\right),$$

$$\frac{\Gamma'(x)}{\Gamma(x)} = -\gamma + \sum_{n=0}^{\infty}(\frac{1}{n+1} - \frac{1}{n+x}) \quad (psifunction) \tag{2}$$

on $\mathbf{C} \setminus \{0, -1, -2, \cdots\}$.

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on $\mathbf{C}\setminus\{0,-1,-2,\cdots\}$.
By (2), $\frac{\Gamma'(z)}{\Gamma(z)}$ maps the open upper half plane Π_+ into itself, namely $\frac{\Gamma'(z)}{\Gamma(z)}$ is a Pick (Nevanlinna) function. $\Gamma'(z)$ does not vanish on Π_+.

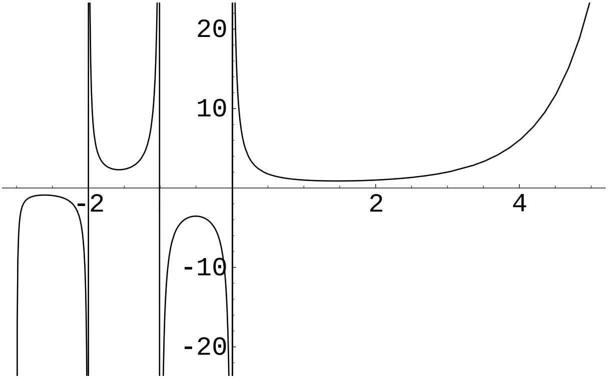

Figure : Gamma function

$$\Gamma(1) = \Gamma(2) = 1,\ \Gamma'(1) = -\gamma,\ \Gamma'(2) = -\gamma + 1.$$

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We call the inverse function of the restriction of $\Gamma(x)$ to (α, ∞) the principal inverse function and write Γ^{-1}.
$\Gamma^{-1}(x)$ is an increasing and concave function defined on $(\Gamma(\alpha), \infty)$.

Main Theorem

Theorem 1

The principal inverse $\Gamma^{-1}(x)$ of $\Gamma(x)$ has the holomorphic extension $\Gamma^{-1}(z)$ to $D := \mathbf{C} \setminus (-\infty, \Gamma(\alpha)]$, which satisfies

(i) $\Gamma^{-1}(\Pi_+) \subset \Pi_+$ and $\Gamma^{-1}(\Pi_-) \subset \Pi_-$,

(ii) $\Gamma^{-1}(z)$ is univalent,

(iii) $\Gamma(\Gamma^{-1}(z)) = z$ for $z \in D$.

Definition 2

Let I be an interval in R and $K(x, y)$ a continuous function defined on $I \times I$. Then $K(x, y)$ is said to be a *positive semidefinite* (p.s.d.) kernel function on an interval $I \times I$ (on I for short) if

$$\iint_{I \times I} K(x, y)\phi(x)\overline{\phi(y)}dxdy \geqq 0 \tag{3}$$

for every complex continuous function ϕ with compact support in I.

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for every complex continuous function ϕ with compact support in I.

- $K(x, y)$ is p.s.d. if and only if for each n and for all n points $x_i \in I$,

$$\sum_{i,j=1}^{n} K(x_i, x_j)z_i\overline{z_j} \geqq 0$$

for n complex numbers z_i.

Definition 3

$K(x, y)$ is said to be *conditionally (or almost) positive semidefinite* (c.p.s.d.) on $I \times I$ (on I for short) if (3) holds for every continuous function ϕ on I such that the support of ϕ is compact and $\int_I \phi(x)dx = 0$.
$K(x, y)$ is said to be *conditionally negative semidefinite* (c.n.s.d.) on I if $-K(x, y)$ is c.p.s.d.

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- $K(x,y)$ is c.p.s.d. if and only if

$$\sum_{i,j=1}^{n} K(x_i, x_j) z_i \overline{z_j} \geqq 0 \tag{4}$$

for each n, for all n points $x_i \in I$ and for n complex numbers z_i with $\sum_{i=1}^{n} z_i = 0$.

Facts

- $K(x,y) = f(x)\overline{f(y)}$ is p.s.d.
- If $K(x,y)$ is p.s.d. on $[a,b]$ and $h:[c,d] \mapsto [a,b]$ is increasing and (differentiable), then so is $K(h(t),h(s))$ on $[c,d]$.
- If $K_t(x,y)$ is p.s.d. for each t, then so is $\int K_t(x,y)d\mu(t)$.
- (Schur) If $K_1(x,y)$ and $K_2(x,y)$ are both p.s.d. on I, then so is the product $K_1(x,y)K_2(x,y)$.
- If $K(x,y)$ is p.s.d. on I, then $K(x,y)$ is c.p.s.d. on I.
- $K(x,y) = x+y$ is not p.s.d. but c. p. s. d. and c. n. s. d. on any I.

Definition 4

Suppose $K(x, y) \geqq 0$ for every x, y in I. Then $K(x, y)$ is said to be *infinitely divisible* on $I \times I$ (on I for short) if $K(x, y)^a$ is positive semi-definite for every $a > 0$.

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is positive semi-definite.

Cauchy kernel

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$$\because \frac{1}{(x+y)^a} = \frac{1}{\Gamma(a)}\int_0^\infty e^{-tx}e^{-ty}t^{a-1}dt$$

Lemma 5

(Fitzgerald, Horn) Let $K(x, y) > 0$ for $x, y \in I$ and suppose $-K(x, y)$ is c.p.s.d. on $I \times I$. Then $\exp(-K(x, y))$ and the reciprocal function $\frac{1}{K(x,y)}$ are infinitely divisible there.

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Example $K(x, y) := x + y$ on $(0, \infty) \times (0, \infty)$
$K(x, y) > 0$ and $-K(x, y)$ is c.p.s.d. on $(0, \infty) \times (0, \infty)$.
$(\exp(-K(x, y)))^a = e^{-ax}e^{-ay}$ is p.s.d. for every $a > 0$.
$\frac{1}{K(x,y)} = \frac{1}{x+y}$ is the Cauchy kernel .

The Löwner kernel

Definition 6

Let $f(x)$ be a real C^1-function on I. Then the *Löwner kernel* function is defined by

$$K_f(x,y) = \begin{cases} \frac{f(x)-f(y)}{x-y} & (x \neq y) \\ f'(x) & (x = y). \end{cases}$$

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Example

(i) For $f(x) = x$, $K_f(x,y) = 1$ is p. s. d. on R^2.

(ii) For $f(x) = -\frac{1}{x+\lambda}$, $K_f(x,y) = \frac{1}{(x+\lambda)(y+\lambda)}$ is p. s. d. on $(-\lambda,\infty)\times(-\lambda,\infty)$ and on $(-\infty,-\lambda)\times(-\infty,-\lambda)$.

(iii) For $f(x) = x^2$, $K_f(x,y) = x+y$ is not p. s. d. but c. p. s. d. and c. n. s. d.

(Löwner Theorem) (also Koranyi)

Let $f(x)$ be a real C^1-function on I. Then the Löwner kernel function $K_f(x, y)$ is p.s.d. on $I \times I$ if and only if $f(x)$ has a holomorphic extension $f(z)$ to Π_+ which is a Pick (Nevanlinna) function.

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We will show

$$K_{\Gamma^{-1}}(x, y)$$

is p. s. d. on $(\Gamma(\alpha), \infty) \times (\Gamma(\alpha), \infty)$.

Lemma 7

(Known result)

$$K_{\log x}(x,y) := \begin{cases} \frac{\log x - \log y}{x-y} & (x \neq y) \\ \frac{1}{x} & (x = y) \end{cases}$$

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Proof. By the formula

$$\log x = \int_0^\infty \left(\frac{-1}{x+t} + \frac{t}{t^2+1}\right)dt \quad (x>0),$$

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$$\log x = \int_0^\infty \left(\frac{-1}{x+t} + \frac{t}{t^2+1}\right)dt \quad (x>0),$$

we obtain

$$K_{\log x}(x,y) = \int_0^\infty \frac{1}{(x+t)(y+t)}dt$$

for $x, y > 0$. □

Lemma 8

$$K_{\log\Gamma(x)}(x,y) := \begin{cases} \frac{\log\Gamma(x)-\log\Gamma(y)}{x-y} & (x \neq y) \\ \frac{\Gamma'(x)}{\Gamma(x)} & (x = y). \end{cases}$$

$-K_{\log\Gamma(x)}(x,y)$ is c.p.s.d. on $(0,\infty)$.

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Proof.

$$\log\Gamma(x) = -\log x - \gamma x + \sum_{n=1}^{\infty}\left(\frac{x}{n} - \log(1+\frac{x}{n})\right)$$

$$g(x) := \sum_{k=1}^{\infty}\left(\frac{x}{k} - \log(1+\frac{x}{k})\right)$$
$$\log\Gamma(x) = -\log x - \gamma x + g(x)$$
$$-K_{\log\Gamma(x)}(x,y) = K_{\log x}(x,y) + \gamma - K_g(x,y).$$

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(i) $-K_{g_n}(x,y)$ is c.p.s.d.

(ii) $g_n'(x) = \sum_{k=1}^{n} \frac{x}{k(k+x)} \Rightarrow \sum_{k=1}^{\infty} \frac{x}{k(k+x)} = g'(x)$ on any finite interval $[0,M]$.

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(i) $-K_{g_n}(x,y)$ is c.p.s.d.

(ii) $g_n'(x) = \sum_{k=1}^{n}\frac{x}{k(k+x)} \Rightarrow \sum_{k=1}^{\infty}\frac{x}{k(k+x)} = g'(x)$ on any finite interval $[0,M]$.

(iii) $K_{g_n}(x,y) - K_g(x,y) = \begin{cases} \frac{1}{x-y}\int_y^x (g_n'(t) - g'(t))\,dt & (x \neq y) \\ g_n'(x) - g'(x) & (x = y) \end{cases}$

Therefore, $-K_g(x,y)$ is c.p.s.d. on $(0,\infty)$. □

Lemma 9

$$\frac{1}{K_{\log \Gamma(x)}(x,y)}$$

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Proof

By the previous lemma

$$K_{\log \Gamma(x)}(x,y) > 0$$

is c.n.s.d. on (α, ∞). By Fitzgerald and Horn's result, we get the required result.

Lemma 10

Let $K_1(x,y)$ be the kernel function defined on $(\alpha,\infty)\times(\alpha,\infty)$ by

$$K_1(x,y)=\begin{cases}\frac{x-y}{\Gamma(x)-\Gamma(y)} & (x\neq y)\\ \frac{1}{\Gamma'(x)} & (x=y).\end{cases}$$

Then $K_1(x,y)$ is p.s.d. on (α,∞).

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Then $K_1(x,y)$ is p.s.d. on (α,∞).

Proof.

$$K_1(x,y)=\begin{cases}\frac{\log\Gamma(x)-\log\Gamma(y)}{\Gamma(x)-\Gamma(y)} & \frac{x-y}{\log\Gamma(x)-\log\Gamma(y)} & (x\neq y)\\ \frac{1}{\Gamma(x)} & \frac{\Gamma(x)}{\Gamma'(x)} & (x=y)\end{cases}$$

$$=K_{\log x}(\Gamma(x),\Gamma(y))\cdot\frac{1}{K_{\log\Gamma(x)}(x,y)}\qquad\square$$

Proof of Theorem

We have shown

$$K_1(x,y) = \begin{cases} \frac{x-y}{\Gamma(x)-\Gamma(y)} & (x \neq y) \\ \frac{1}{\Gamma'(x)} & (x = y). \end{cases}$$

is p.s.d. on (α, ∞). Hence

$$K_{\Gamma^{-1}}(x,y) = \begin{cases} \frac{\Gamma^{-1}(x)-\Gamma^{-1}(y)}{x-y} & (x \neq y) \\ (\Gamma^{-1})'(x) & (x = y) \end{cases} = K_1(\Gamma^{-1}(x), \Gamma^{-1}(y))$$

is p.s.d. on $(\Gamma(\alpha), \infty) \times (\Gamma(\alpha), \infty)$.
Thus by the Löwner theorem, $\Gamma^{-1}(x)$ has the holomorphic extension $\Gamma^{-1}(z)$ onto Π_+, which is a Pick function.
By reflection $\Gamma^{-1}(x)$ has also holomorphic extension to Π_- and the range is in it.

$\Gamma(\Gamma^{-1}(z))$ is thus holomorphic on the simply connected domain $D := \mathbf{C} \setminus (-\infty, \Gamma(\alpha)]$, and $\Gamma(\Gamma^{-1}(x)) = x$ for $\Gamma(\alpha) < x < \infty$. By the unicity theorem, $\Gamma\left(\Gamma^{-1}(z)\right) = z$ for $z \in D$. It is clear that $\Gamma^{-1}(z)$ is univalent. □

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Corollary 11

There is a Borel measure μ so that

$$\Gamma^{-1}(x) = a + bx + \int_{-\infty}^{\Gamma(\alpha)} (\frac{1}{t-x} - \frac{t}{t^2+1}) d\mu(t), \tag{5}$$

where $\int_{-\infty}^{\Gamma(\alpha)} \frac{1}{t^2+1} d\mu(t) < \infty$, and a, b are real numbers and $b \geqq 0$.

Matrix inequality

Theorem 12

The principal inverse $\Gamma^{-1}(x)$ of $\Gamma(x)$ is operator monotone on $[\Gamma(\alpha), \infty)$; i.e.,
and hence for bounded self-adjoint operators A, B whose spectra are in $[\Gamma(\alpha), \infty)$

$$A \leqq B \quad \Rightarrow \Gamma^{-1}(A) \leqq \Gamma^{-1}(B).$$

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$$A \leqq B \quad \Rightarrow \Gamma^{-1}(A) \leqq \Gamma^{-1}(B).$$

Proof $A \leqq B$ implies that $-(A - tI)^{-1} \leqq -(B - tI)^{-1}$ for $t < \Gamma(\alpha)$. From

$$\Gamma^{-1}(x) = a + bx + \int_{-\infty}^{\Gamma(\alpha)} \left(\frac{-1}{x-t} - \frac{t}{t^2+1}\right) d\mu(t)$$

we have $\Gamma^{-1}(A) \leqq \Gamma^{-1}(B)$. □

$$K_2(x,y) := K_{\log\Gamma(x)}(x,y) = \begin{cases} \frac{\log\Gamma(x)-\log\Gamma(y)}{x-y} & (x \neq y) \\ \frac{\Gamma'(x)}{\Gamma(x)} & (x = y). \end{cases}$$

Then

$$e^{-K_2(x,y)} = \begin{cases} \left(\frac{\Gamma(y)}{\Gamma(x)}\right)^{\frac{1}{x-y}} & (x \neq y) \\ e^{-\frac{\Gamma'(x)}{\Gamma(x)}} & (x = y) \end{cases}$$

is infinitely divisible. Since $\Gamma(x+1) = x\Gamma(x)$,

$$\frac{\Gamma'(1)}{\Gamma(1)} = -\gamma,\ \frac{\Gamma'(m+1)}{\Gamma(m+1)} = \frac{\Gamma'(m)}{\Gamma(m)} + \frac{1}{m},\ \frac{\Gamma(n)}{\Gamma(m)} = \frac{(n-1)!}{(m-1)!}.$$

matrix

The following $(n+1)\times(n+1)$ matrix is therefore not only p.s.d. but also infinitely divisible.

$$\left(e^{-K_2(i,j)}\right) =$$

$$= \begin{pmatrix} e^{\gamma} & (\frac{1!}{1!})^{-1} & (2!)^{-\frac{1}{2}} & (3!)^{-\frac{1}{3}} & \cdots & (n!)^{-\frac{1}{n}} \\ (\frac{1!}{1!})^{-1} & e^{\gamma-1} & (\frac{2!}{1!})^{-1} & (\frac{3!}{1!})^{-\frac{1}{2}} & \cdots & (\frac{n!}{1!})^{-\frac{1}{n-1}} \\ (2!)^{-\frac{1}{2}} & (\frac{2!}{1!})^{-1} & e^{\gamma-1-\frac{1}{2}} & (\frac{3!}{2!})^{-1} & \cdots & (\frac{n!}{2!})^{-\frac{1}{n-2}} \\ (3!)^{-\frac{1}{3}} & (\frac{3!}{1!})^{-\frac{1}{2}} & (\frac{3!}{2!})^{-1} & e^{\gamma-1-\frac{1}{2}-\frac{1}{3}} & \cdots & (\frac{n!}{3!})^{-\frac{1}{n-3}} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ (n!)^{-\frac{1}{n}} & (\frac{n!}{1!})^{-\frac{1}{n-1}} & (\frac{n!}{2!})^{-\frac{1}{n-2}} & (\frac{n!}{3!})^{-\frac{1}{n-3}} & \cdots & e^{\gamma-1-\frac{1}{2}-\cdots-\frac{1}{n}} \end{pmatrix}$$

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