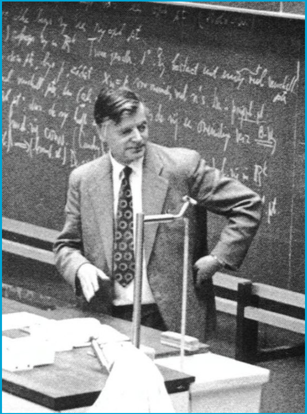
γ-Radonifying operators

Jan van Neerven



Positivity VII (Zaanen Centennial Conference), Leiden, July 22–26, 2013

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Adriaan Cornelis Zaanen

Biography MathSciNet

Ph.D. Universiteit Leiden 1938

Dissertation: *Over reeksen van eigenfuncties van zekere randproblemen*

Advisor: Johannes Droste

Students:
Click here to see the students ordered by family name.

Name	School	Year	Descendants
Wilhelmus Luxemburg	Technische Universiteit Delft	1955	40
Barend Strydom	Universiteit Leiden	1959	
Marinus Kaashoek	Universiteit Leiden	1964	25
Adrianus van Eynsbergen	Universiteit Leiden	1967	
Jacobus Grobler	Universiteit Leiden	1970	11
Nicolaas van Arkel	Universiteit Leiden	1970	
Ebbel de Jonge	Universiteit Leiden	1973	
Charles Huijsmans	Universiteit Leiden	1973	2
Pieter Maritz	Universiteit Leiden	1975	
Willem Claas	Universiteit Leiden	1977	
Anton Schep	Universiteit Leiden	1977	1
Pieter Kosterse	Universiteit Leiden	1978	
Willem Vietsch	Universiteit Leiden	1979	
Bernardus de Pagter	Universiteit Leiden	1981	9

According to our current on-line database, Adriaan Zaanen has 14 students and 100 descendants.
We welcome any additional information.

If you have additional information or corrections regarding this mathematician, please use the update form.
To submit students of this mathematician, please use the new data form.

Math. Ann. 283, 511–518 (1989)

Mathematische Annalen

A Characterization of Sun-Reflexivity *

B. de Pagter

Department of Mathematics, Delft University of Technology, Julianalaan 132, 2628 BL Delft, The Netherlands

1. Introduction

The duality theory for strongly continuous semigroups of bounded linear operators (i.e., C_0-semigroups) in a Banach space was initiated by Phillips in [8]. One of the difficulties in dealing with adjoint semigroups is that the adjoint semigroup $\{T^*(t)\}_{t \geqq 0}$ of a C_0-semigroup $\{T(t)\}_{t \geqq 0}$ in a Banach space X, need not be strongly continuous in X^*. However, restricting $\{T^*(t)\}_{t \geqq 0}$ to the closed subspace $X^{\odot}$ of X^* on which $\{T^*(t)\}_{t \geqq 0}$ is strongly continuous, we obtain a C_0-

Overview of the talk

1. γ-Radonifying operators
2. Main examples
3. Properties
4. The extension problem
5. Applications

1. γ-Radonifying operators

H – Hilbert space
X – Banach space

On the algebraic tensor product $H \otimes X$ we introduce the norm

$$\Big\| \sum_{n=1}^{N} h_n \otimes x_n \Big\|_{\gamma(H,X)} := \Big(\mathbb{E} \Big\| \sum_{n=1}^{N} \gamma_n \otimes x_n \Big\|^2 \Big)^{1/2}$$

where

- $h_1, \dots, h_N$ orthonormal in H,
- $\gamma_1, \dots, \gamma_N$ a Gaussian sequence.

Well-definedness follows from the orthogonal invariance of Gaussian sequences.

Definition. $\gamma(H, X)$ – completion of $H \otimes X$ with respect to $\| \cdot \|_{\gamma(H,X)}$.

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Remark. By the Kahane-Khintchine inequalities, for each $p \in [1, \infty)$

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Proposition. *The identity mapping* $h \otimes x \mapsto h \otimes x$ *extends to a contractive embedding*

$$\gamma(H, X) \hookrightarrow \mathscr{L}(H, X)$$

Thus we may identify each element of $T \in \gamma(H, X)$ in a unique way with a bounded linear operator $T : H \to X$.

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$$\phi = \sum_{n=1}^{N} \frac{\mathbf{1}_{(t_{n-1},t_n)}}{\sqrt{t_n - t_{n-1}}} \otimes x_n$$

we have

$$\mathbb{E}\Big\| \int_0^\infty \phi \, dW \Big\|^p \stackrel{\text{def}}{=} \mathbb{E}\Big\| \sum_{n=1}^{N} \frac{W(t_n) - W(t_{n-1})}{\sqrt{t_n - t_{n-1}}} \otimes x_n \Big\|^p$$

$$= \mathbb{E}\Big\| \sum_{n=1}^{N} \gamma_n \otimes x_n \Big\|^p = \|\phi\|^p_{\gamma^p(H,X)}.$$

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More generally, suppose that $W : H \to L^2(\Omega)$ is a Gaussian process satisfying

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2. Main examples

Example 1. If X is a Hilbert space, then for $T = \sum_{n=1}^{N} h_n \otimes x_n$,

$$\|T\|_{\gamma(H,X)}^2 = \mathbb{E}\Big\|\sum_{n=1}^{N} \gamma_n \otimes x_n\Big\|^2 = \mathbb{E}\sum_{m,n=1}^{N} \gamma_m\gamma_n[x_m, x_n]$$

$$= \sum_{n=1}^{N} \|x_n\|^2 = \sum_{n=1}^{N} \|Th_n\|^2 = \|T\|_{\mathscr{L}^2(H,X)}^2.$$

Thus

$$\boxed{\gamma(H,X) = \mathscr{L}^2(H,X).}$$

In particular,

$$\gamma(L^2(\mu),X) = \mathscr{L}^2(L^2(\mu),X) = L^2(\mu;X)$$

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Example 2. Let (S,μ) be a measure space. If $X = L^p(\mu)$, then

$$T_\phi : h \mapsto [\phi, h]$$

establishes an isomorphism of Banach spaces

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and, by the Khintchine inequalities,

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Example 3. (Wiener measure) The indefinite integral

$$(Jf)(t) := \int_0^t f(s)\,ds$$

is γ-radonifying from $L^2(0,1)$ to $C^\alpha[0,1]$ for all $0 \leqslant \alpha < \frac{1}{2}$.

Proof. (Ciesielski)

- If H is separable with ONB $(h_n)_{n=1}^\infty$, then

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Example 4. (Regularity implies γ-radonification) If $\phi : (0,1) \to X$ is continuously differentiable and

$$\int_0^1 \sqrt{s}\|\phi'(s)\|\,ds < \infty,$$

then $\phi \in \gamma(L^2(0,1), X)$ and

$$\|\phi\|_{\gamma(L^2(0,1),X)} \leqslant \|\phi(1)\| + \int_0^1 \sqrt{s}\|\phi'(s)\|\,ds.$$

3. Properties

For real numbers a and b,

$$a \leqslant b \iff \exists c \geqslant 0 : a + c = b.$$

Ah! Positivity at last!

The Gaussian version of this reads as follows:

Lemma. If, for all $x^* \in X^*$,

$$\sum_{m=1}^{M} |\langle x_m, x^* \rangle|^2 \leqslant \sum_{n=1}^{N} |\langle y_n, x^* \rangle|^2,$$

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Proof: Set

$$\xi = \sum_{m=1}^{M} \gamma_m x_m, \quad \eta = \sum_{n=1}^{N} \gamma_n y_n.$$

By the assumption,

$$\xi \leqslant \eta$$

in the sense of covariances, i.e., $C_\eta - C_\xi$ is positive definite and symmetric. Hence there exists

$$\zeta = \sum_{m=M+1}^{K} \gamma_m z_m$$

such that

$$\xi + \zeta = \eta.$$

Then, by independence and symmetry,

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Corollary. (Ideal property) *For all* $p \in [1,\infty)$,

$$\|UTS\|_{\gamma^p(H,X)} \leqslant \|U\|_{\mathscr{L}(X)}\|T\|_{\gamma^p(H,X)}\|S\|_{\mathscr{L}(H)}.$$

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Corollary. (Domination) *Let* H_1 *and* H_2 *be Hilbert spaces and let* $T_1 \in \mathscr{L}(H_1, X)$ *and* $T_2 \in \mathscr{L}(H_2, X)$*. If*

$$\| T_1^* x^* \|_{H_1^*} \leqslant \| T_2^* x^* \|_{H_2^*} \qquad \forall x^* \in X^*,$$

then $T_2 \in \gamma(H_2, X)$ *implies* $T_1 \in \gamma(H_1, X)$ *and for all* $1 \leqslant p < \infty$ *we have*

$$\| T_1 \|_{\gamma^p(H_1, X)} \leqslant \| T_2 \|_{\gamma^p(H_2, X)}.$$

Proposition. (γ-Fatou lemma) Assume that $c_0 \not\subseteq X$. *Let* $(T_n)_{n\geqslant 1}$ *be a bounded sequence in* $\gamma(H,X)$. *If*

$$\lim_{n\to\infty} \langle T_n h, x^*\rangle = \langle Th, x^*\rangle \quad \forall h \in H,\ x^* \in X^*,$$

then $T \in \gamma(H,X)$ *and for all* $1 \leqslant p < \infty$ *we have*

$$\| T\|_{\gamma^p(H,X)} \leqslant \liminf_{n\to\infty} \| T_n\|_{\gamma^p(H,X)}.$$

Proposition. (Dominated convergence) *Suppose that* $T_n, T \in \mathscr{L}(H, X)$ *satisfy* $\lim_{n\to\infty} T_n^* x^* = T^* x^*$ *in* H^* *for all* $x^* \in X^*$. *If there exists* $U \in \gamma(H, X)$ *such that*

$$\|T_n^* x^*\|_{H^*} \leqslant \|U^* x^*\|_{H^*}$$

for all $n \geqslant 1$ *and* $x^* \in X^*$, *then* $T_n, T \in \gamma(H, X)$ *and*

$$\lim_{n\to\infty} \|T_n - T\|_{\gamma(H,X)} = 0.$$

Proposition. (Fubini) *For all* $p \in [1, \infty)$, *the mapping*

$$h_1 \otimes (h_2 \otimes x) \mapsto h_2 \otimes (h_1 \otimes x)$$

extends to an isometry of Banach spaces

$$\gamma^p(H_1, \gamma^p(H_2, X)) = \gamma^p(H_2, \gamma^p(H_1, X))$$

Proposition. (Trace duality) *For all* $T \in \gamma(H,X)$ *and* $U \in \gamma(H,X^*)$ *we have* $U^*T \in \mathscr{L}^1(H)$ *and*

$$|\mathrm{tr}(U^*T)| \leqslant \|U^*T\|_{\mathscr{L}^1(H)} \leqslant \|T\|_{\gamma(H,X)}\|U\|_{\gamma(H,X^*)}.$$

As a consequence, $\gamma(H,X^*) \hookrightarrow (\gamma(H,X))^*$ contractively.
If X is K-convex, this induces an isomorphism of Banach spaces

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Let (S, μ) be a measure space.

Proposition. (Multipliers, Kalton-Weis) *Suppose that* $M : S \to \mathscr{L}(X, Y)$ *has* γ*-bounded range. If* $c_0 \not\subseteq Y$*, then for all* μ*-simple* $\phi : S \to X$ *and* $p \in [1, \infty)$ *we have*

$$\|M\phi\|_{\gamma^p(L^2(S),Y)} \leqslant \gamma^p(M)\,\|\phi\|_{\gamma^p(L^2(S),X)}.$$

As a consequence, the mapping $\widetilde{M} : \phi \mapsto M\phi$ has a unique extension to a bounded operator

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4. The extension problem

If $T : L^p(\mu_1) \to L^p(\mu_2)$ is a bounded linear operator, does $T \otimes I_X$ extend to a bounded operator $L^p(\mu_1; X) \to L^p(\mu_2; X)$?

Yes, in special circumstances:

- if $p = 1$
- if T is positive
- if X is a Hilbert space (Paley, Marcinkiewicz & Zygmund)

No, in general:

- FT bounded on $L^2(\mathbb{R}; X) \iff X \simeq$ Hilbert space (Kwapień)
- HT bounded on $L^p(\mathbb{R}; X) \iff X$ UMD (Burkholder, Bourgain)
- Itô integral bounded from $L^2(\mathbb{R}_+; X)$ to $L^2(\Omega; X) \iff X$ type 2 (Hoffmann-Jørgensen & Pisier, Rosiński & Suchanecki)

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$$\widetilde{T} : h \otimes x \mapsto Th \otimes x, \quad h \in H_1,\ x \in X,$$

has a unique extension to a bounded linear operator

$$\widetilde{T} : \gamma(H_1, X) \to \gamma(H_2, X)$$

of the same norm.

5. Applications

The Itô isometry for step functions $\phi : \mathbb{R}_+ \to X$ extends to adapted step processes $\phi : \mathbb{R}_+ \times \Omega \to X$ as follows:

Theorem. (Itô isomorphism, vN-Veraar-Weis, AOP 2007) *Let X be a UMD space and let $p \in (1, \infty)$. For all adapted step processes $\phi : \mathbb{R}_+ \times \Omega \to X$ we have*

$$\frac{1}{C^p} \|\phi\|^p_{L^p(\Omega;\gamma^p(L^2(\mathbb{R}_+),X))} \leqslant \mathbb{E}\Big\| \int_0^\infty \phi \, dW \Big\|^p \leqslant C^p \|\phi\|^p_{L^p(\Omega;\gamma^p(L^2(\mathbb{R}_+),X))}$$

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Let $(p, q) \in (2, \infty) \times [2, \infty)$ or $p = q = 2$.

Let $D \subseteq \mathbb{R}^d$ be open and let A admit a bounded H^∞-calculus of angle less than $\pi/2$ on $L^q(D)$.

Theorem. (Maximal L^p-regularity, vN-Veraar-Weis, AOP 2012) *For all adapted $G \in L^p(\mathbb{R}_+ \times \Omega; L^q(D))$, the convolution process*

$$U(t) = \int_0^t e^{-(t-s)A} G(s)\, dW(s)$$

satisfies the stochastic maximal L^p-regularity estimate

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Proof.

- two-sided estimate for the UMD stochastic integral
- γ-boundedness techniques
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