Tensor products in Riesz space theory as quotients of free spaces

Jan van Waaij

jvanwaaij@gmail.com

joint work with Onno van Gaans and Marcel de Jeu

August 1, 2013

Mathematical Institute, Leiden University

Contents

1 Preface

There is a construction of the tensor product of Riesz spaces due to B. de Pagter as a quotient of a free Riesz space over a suitable chosen set. In my master thesis 'Tensor products in Riesz space theory' (Leiden University, supervisors: Onno van Gaans and Marcel de Jeu) I give new constructions for the tensor product of integrally closed directed partially ordered vector spaces and for the tensor product of Banach lattices as quotients of free Riesz spaces and free Banach lattices respectively. The main results are in chapters 5 and 6.

2 Preliminaries

Definition 2.1. Let E be a partially ordered vector space. Then E is Archimedean if $nx \leq y$ for all $n \in \mathbb{Z}$ implies that $x = 0$, for all $x, y \in E$. We say that E is integrally closed if $nx \leq y$ for all $n \in \mathbb{N}_0$ implies that $x \leq 0$, for all $x, y \in E$.

Remark 2.2. Every integrally closed partially ordered vector space is Archimedean. The converse is not true. In a Riesz space the notions are equivalent.

For a non-empty set $X \subset E$ we denote by X^u the set of all upper bounds of X and by X^l the set of all lower bounds of X.

Definition 2.3. Let E be a partially ordered vector space. Then E is a pre-Riesz space if E is directed and $(x + X)^u \subset X^u$ implies that $x \geq 0$ for all $x \in E$ and for all non-empty finite sets $X \subset E$.

Remark 2.4. Every Riesz space is pre-Riesz and every directed integrally closed partially ordered vector space is pre-Riesz.

Theorem 2.5 (Van Haandel)**.** *For every pre-Riesz space E there is an essentially unique Riesz space E^r and a bipositive linear map $\varphi_E : E \to E^r$ such that $\varphi_E(E)$ is order dense in E^r and generates E^r as a Riesz space. We say that (E^r, φ_E) is the Riesz completion of E. Suppose (L, ϕ) and (M, ψ) are Riesz completions of E then there is a unique Riesz homomorphism $f : L \to M$ with $\psi = f \circ \phi$.*

In a Riesz space L, an ideal is a subspace $I \subset L$ such that if $y \in I$ and x is in L such that $|x| \leq |y|$, then $x \in I$ [1, page 25]. A good definition for general partially ordered vector spaces is difficult. Maybe the best choice is the following definition. It coincides with the definition of an ideal in a Riesz space. For a comprehensive discussion see [8].

Definition 2.6. Let E be a partially ordered vector space. An ideal I of E is a subspace of E that is directed and it has the following property: for every $y \in I$ and for every $x \in E$ such that $\{-x, x\}^u \supseteq \{-y, y\}^u$ one has that $x \in I$.

Remark 2.7. If E is a Riesz space, then $\{-x, x\}^u = \{|x|\}^u \supseteq \{-y, y\}^u = \{|y|\}^u$ if and only if $|x| \leq |y|$. Clearly every Riesz ideal is directed. Thus Definition 2.6 coincides with the usual definition of a Riesz ideal.

Proposition 2.8. *Let E be a partially ordered vector space and I an ideal of E. For every $y \in I$, we have that $[-y, y] \subset I$.*

Proof. Let E be a partially ordered vector space and let I an ideal of E. Let $y \in I$ and suppose that $[-y, y] \neq \emptyset$. Let $x \in [-y, y]$. We have to show that $x \in I$. Note that $x \leq y$ and that $-x \leq y$. Let $z \in \{-y, y\}^u$, then $z \geq y \geq \{\pm x\}$. Thus $z \in \{\pm x\}^u$. It follows that $\{\pm y\}^u \subseteq \{\pm x\}^u$. Thus $x \in I$. □

Remark 2.9. From Proposition 2.8 follows that our ideal is solid in the sense of [5, Definition 351 I]: a subset A of E is solid if $A = \bigcup_{x \in A} [-x, x]$.

Definition 2.10. Let E be a partially ordered vector space. A sequence $\{x_n\}_{n\geq 1}$ converges relatively uniformly (ru-converges) to $x \in E$, if there exist a $u \in E^+$ and a sequence of positive real numbers $\{\varepsilon_n\}_{n\geq 1}$ with $\varepsilon_n \downarrow 0$ such that $-\varepsilon_n u \leq x_n - x \leq \varepsilon_n u$, for all $n \in \mathbb{N}$. A set $A \subset E$ is relatively uniformly closed (ru-closed) if for every sequence $\{x_n\}_{n\geq 1} \subset A$ that is relatively uniformly convergent to an $x \in L$, we have that $x \in A$. Clearly, the intersection of an arbitrary collection of relatively uniformly closed sets if relatively uniformly closed. We define the relatively uniformly closure (ru-closure) of a set A to be the intersection of all relatively uniformly closed sets B that contain A. This collection is not empty since it contains E.

Theorem 2.11. *Let E be a partially ordered vector space. Let I be an ideal of E. If E/I is Archimedean, then I is relatively uniformly closed. If E is a Riesz space, then E/I is Archimedean if and only if I is relatively uniformly closed.*

Proof. Let E be a partially ordered vector space and let I be an ideal of E. Let $q : E \to E/I$ be the canonical quotient map. Suppose E/I is Archimedean. Let $\{x_n\}_{n\geq 1}$ be a sequence in I that relative uniform converges to $x \in E$. By definition, there is a sequence of positive real numbers $\{\varepsilon_n\}_{n\geq 1}$ and a $u \in E^+$ such that $\varepsilon_n \downarrow 0$ and $-\varepsilon_n u \leq x_n - x \leq \varepsilon_n u$, for all $n \in \mathbb{N}$. Thus $-\varepsilon_n q(u) \leq -q(x) \leq \varepsilon_n q(u)$, for all $n \in \mathbb{N}$. For every $m \in \mathbb{N}$ there is an $n_m \in \mathbb{N}$ such that $\frac{1}{m} \geq \varepsilon_{n_m}$, so $\frac{1}{m} q(u) \leq -q(x) \leq \frac{1}{m} q(u)$, for all $m \in \mathbb{N}$. It follows that $nq(x) \leq q(u)$, for all $n \in \mathbb{Z}$. Since E/I is Archimedean we have that $q(x) = 0$. Thus $x \in I$ and I is relatively uniformly closed.
Assume further that E is a Riesz space. Suppose that I is relatively uniformly closed. Let $x, y \in E$ be such that $n[x] \leq [y]$, for all $n \in \mathbb{Z}$, then $[y] \geq 0$. Therefore we may assume that $y \geq 0$. We have for all $n \in \mathbb{N}$,

$$-\tfrac{1}{n}[y] \leq [x] \leq \tfrac{1}{n}[y].$$

Thus there are $i_n, i'_n \in I$ such that

$$x + i_n \leq \tfrac{1}{n} y,$$

and

$$-x + i'_n \leq \tfrac{1}{n} y,$$

for all $n \in \mathbb{N}$. Thus for $x_n = (x + i_n) \vee (-x + i'_n) \vee 0 \in |x| + I \subset E$ we have

$$-\tfrac{1}{n} y \leq 0 \leq x_n - 0 \leq \tfrac{1}{n} y.$$

Hence $\{x_n\}_{n=1}^{\infty}$ is a sequence in $|x| + I$ that converges relatively uniformly to 0. We have that $|x| + I$ is ru-closed hence $0 \in |x| + I$. Thus $[x] = 0$. We conclude that E/I is Archimedean. ☐

Theorem 2.12. *Let E and F be partially ordered vector spaces with F Archimedean. Let $\phi : E \to F$ be positive. Then $\ker \phi$ is ru-closed.*

Proof. Let E and F be partially ordered vector spaces with F Archimedean. Let $\phi : E \to F$ be positive. Let $\{x_n\}_{n=1}^{\infty}$ be a sequence in $\ker \phi$ that is ru-convergent to $x \in E$. So there exists a real sequence $\{\varepsilon_n\}_{n=1}^{\infty}, \varepsilon_n \downarrow 0$ and a $u \in E^+$, such that

$$-\varepsilon_n u \leq x_n - x \leq \varepsilon_n u,$$

for all $n \in \mathbb{N}$. Thus

$$-\varepsilon_n \phi(u) \leq \phi(x) \leq \varepsilon_n \phi(u),$$

for all $n \in \mathbb{N}$. For every $m \in \mathbb{N}$, there is an $n_m \in \mathbb{N}$, such that $\varepsilon_{n_m} \leq \frac{1}{m}$. Thus

$$-\tfrac{1}{n} \phi(u) \leq \phi(x) \leq \tfrac{1}{n} \phi(u),$$

for all $n \in \mathbb{N}$. Furthermore $\phi(u) \geq 0$, so $n\phi(x) \leq \phi(u)$, for all $n \in \mathbb{Z}$. Since F is Archimedean, we have that $\phi(x) = 0$, that is $x \in \ker \phi$. We conclude that $\ker \phi$ is ru-closed. ☐

Definition 2.13. Let L be a Riesz space. A Riesz norm or lattice norm $||\cdot||$ on L is a norm on L such that if $|x| \leq |y|$ then $||x|| \leq ||y||$, for all $x, y \in L$. The pair $(L, ||\cdot||)$ is called a normed Riesz space. If the induced metric on L through $||\cdot||$ is complete, then we call $(L, ||\cdot||)$ a Banach lattice.

Definition 2.14. Let L, M and N be Riesz spaces. A Riesz bimorphism is a bilinear map $\phi : L \times M \to N$ such that for all $x \in L^+ : \phi(x, \cdot)$ is a Riesz homomorphism and for all $y \in M^+ : \phi(\cdot, y)$ is a Riesz homomorphism.

Proposition 2.15. *Let L, M and N be Riesz spaces and $\phi : L \times M \to N$ be bilinear. Then ϕ is a Riesz bimorphism if and only if for all $x \in L$ and $y \in M$ we have $|\phi(x, y)| = \phi(|x|, |y|)$.*

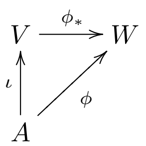
3 Free spaces

The free vector space over a set A are all linear combinations of elements of A. Thus A is the vector space basis for the free vector space. Something similar can be done for Riesz spaces, normed Riesz spaces and Banach lattices. We define and study these objects here. We need the free spaces for the construction of various tensor products.
Let B be a set and let $A \subset B$. We define $r_A : \mathbb{R}^B \to \mathbb{R}^A$ to be the restriction map, thus for $f \in \mathbb{R}^B$ we define $r_A(f) = f|_A$. It is clear that r_A is a surjective Riesz homomorphism. Sometimes we write $\xi_A = r_A(\xi)$ for $\xi \in \mathbb{R}^B$. We define $j_A : \mathbb{R}^{\mathbb{R}^A} \to \mathbb{R}^{\mathbb{R}^B}$ by $j_A(f)(\xi) = f(\xi_A) = f(r_A(\xi)) = f(\xi|_A)$, where $f \in \mathbb{R}^{\mathbb{R}^A}$ and $\xi \in \mathbb{R}^B$. Then j_A is an injective Riesz homomorphism and hence bipositive.

3.1 Free vector spaces and free Riesz spaces

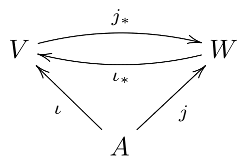
In this subsection we study the free vector space and the free Riesz space and we will show that the free vector space is a subspace of the free Riesz space.

Definition 3.1. A *free vector space* over a set A is a pair (V, ι) where $\iota : A \to V$ is a map and V is a real vector space, such that for every real vector space W and for every map $\phi : A \to W$ there is a unique linear map $\phi_* : V \to W$ such that $\phi = \phi_* \circ \iota$.



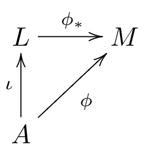
Lemma 3.2. *Let A be a set. Suppose (V, ι) and (W, j) are free vector spaces over A. Then there is a unique bijective linear map $\phi : V \to W$ such that $j = \phi \circ \iota$.*

Proof. Let A be a set. Suppose that (V, ι) and (W, j) are free vector spaces over A. By definition there are unique linear maps $\iota_* : W \to V$ and $j_* : V \to W$ such that $\iota = \iota_* \circ j$ and $j = j_* \circ \iota$. Thus $\iota = \iota_* \circ j_* \circ \iota$. Note that also the identity map id_V on V is a linear map such that $\iota = id_V \circ \iota$. From the uniqueness statement it follows that $id_V = \iota_* \circ j_*$. Similarly, we have that the identity map id_W on W is equal to $j_* \circ \iota_*$. Define $\phi = j_* : V \to W$. Then ϕ is the unique isomorphism $\psi : V \to W$ such that $j = \psi \circ \iota$.



□

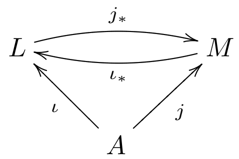
Definition 3.3. A free Riesz space over a set A is a pair (L, ι) where $\iota : A \to L$ is a map and L is a Riesz space such that for every Riesz space M and every map $\phi : A \to M$ there is a unique Riesz homomorphism $\phi_* : L \to M$ such that $\phi = \phi_* \circ \iota$.



The next lemma can be found in [14, Proposition 3.3].

Lemma 3.4. *A free Riesz space is unique if it exists, in the following sense: let (L, ι) and (M, j) be two free Riesz spaces over a set A, then there is a unique (surjective) Riesz isomorphism $T : L \to M$ such that $T \circ \iota = j$. In particular T is an order isomorphism.*

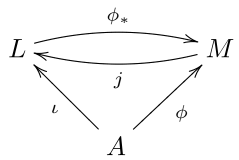
Proof. Let A be a set and suppose that (L, ι) and (M, j) are two free Riesz spaces over A. There is a unique Riesz homomorphism $j_* : L \to M$ such that $j = j_* \circ \iota$ and a unique Riesz homomorphism $\iota_* : M \to L$ such that $\iota = \iota_* \circ j$. Note that $\iota = \iota_* \circ j = \iota_* \circ j_* \circ j$ and $\iota_* \circ j_* : L \to L$ is a Riesz homomorphism. Note that also that the identity map id_L on L is a Riesz homomorphism such that $\iota = id_L \circ \iota$. From the uniqueness statement follows that $id_L = \iota_* \circ j_*$. Likewise is the identity map id_M on M satisfies $id_M = j_* \circ \iota_*$. Define $T = j_* : L \to M$, then T is an invertible Riesz homomorphism and $T^{-1} = \iota_*$ is a Riesz homomorphism and $T \circ \iota = j_* \circ \iota = j$. Moreover T is the unique Riesz homomorphism with this properties. In particular T is an order isomorphism.



□

Theorem 3.5. *([14, Proposition 3.2]). If (L, ι) is a free Riesz space over a set A, then L is generated as Riesz space by $\iota(A)$.*

Proof. Let (L, ι) be a free Riesz space over a set A. Let M be the Riesz subspace of L generated by $\iota(A)$. Define the map $\phi : A \to M$ by $\phi(a) = \iota(a)$. By definition, there is a Riesz homomorphism $\phi_* : L \to M$ such that $\phi = \phi_* \circ \iota$. Let $j : M \to L$ be the inclusion map. Then $j \circ \phi_* : L \to L$ satisfies $j \circ \phi_* \circ \iota = \iota$. By definition, the identity map on L, id_L is the unique Riesz homomorphism $\psi : L \to L$ that satisfies $\iota = \psi \circ \iota$. Thus $j \circ \phi_* = id_L$. But that implies that j is surjective, so $M = L$. We conclude that L is generated as Riesz space by $\iota(A)$.



□

For every set the the free Riesz space exists, see [14, Proposition 3.7]. The case $A = \emptyset$ is trivial.

Theorem 3.6. *Let A be a set. If $A = \emptyset$, then $(0, \emptyset)$ is the free Riesz space over A. If $A \neq \emptyset$ then $(FRS(A), \iota)$ is the free Riesz space over A, where $FRS(A)$ is the Riesz subspace of $\mathbb{R}^{\mathbb{R}^A}$ generated by elements $\xi_a \in \mathbb{R}^{\mathbb{R}^A}$, defined by $\xi_a(f) = f(a)$ for $a \in A$ and $f \in \mathbb{R}^A$, and where $\iota : A \to FRS(A)$ is defined by $a \mapsto \xi_a$. Moreover ι is injective and $FRS(A)$ is Archimedean.*

Lemma 3.7. *Let A be a set. Let ι be as in Theorem 3.6. Then $\iota(A)$ is a linearly independent set.*

Proof. The case $A = \emptyset$ is trivial, so suppose that $A \neq \emptyset$. Let $a_1, \ldots, a_n \in A$ be finitely many mutually different elements. Let $\lambda_1, \ldots, \lambda_n \in \mathbb{R}$ be such that $\sum_{i=1}^n \lambda_i \xi_{a_i} = 0$. Then, for all $f \in \mathbb{R}^A$ we have that

$$\left(\sum_{i=1}^n \lambda_i \xi_{a_i} \right)(f) = \sum_{i=1}^n \lambda_i f(a_i) = 0.$$

Define $f_k \in \mathbb{R}^A$ by $f(a_k) = 1$ and $f(a) = 0$ for $a \in A \backslash \{a_k\}$, where $k \in \{1, \ldots, n\}$. Then

$$\left(\sum_{i=1}^n \lambda_i \xi_{a_i} \right)(f_k) = \sum_{i=1}^n \lambda_i f_k(a_i) = \lambda_k = 0.$$

Thus $\lambda_k = 0$, for $k \in \{1 \dots, n\}$. It follows that $\xi_1, \dots, \xi_n$ are linearly independent. □

Theorem 3.8. *Let A be a set. If $A = \emptyset$, then $(0, \emptyset)$ is the free vector space over A. If $A \neq \emptyset$, let ξ_a be as in Theorem 3.6 on the previous page, where $a \in A$. Let $FVS(A) = Span\{\xi_a : a \in A\} \subset FRS(A) \subset \mathbb{R}^{\mathbb{R}^A}$. Define $\iota : A \to FVS(A)$ by $a \mapsto \xi_a$, where $a \in A$. Then $(FVS(A), \iota)$ is the free vector space over A.*

Proof. Let A be a set. The statement is trivial if $A = \emptyset$. So suppose that A has at least one element. Let $\mathrm{FVS}(A)$ and ι be as in the statement of the theorem. By Lemma 3.7 on the preceding page $\{\xi_a : a \in A\}$ is a linearly independent set. Let W be an arbitrary vector space and let $\phi : A \to W$ be a map. Define a linear map $\phi_* : \mathrm{FVS}(A) \to W$ on its basis elements ξ_a by $\xi_a \mapsto \phi(a)$, where $a \in A$. Thus $\phi = \phi_* \circ \iota$. For any other linear map $\psi : \mathrm{FVS}(A) \to W$ that satisfies $\phi = \psi \circ \iota$, we have that $\psi(\xi_a) = \phi(a) = \phi_*(\xi_a)$. Since the ξ_a generate $\mathrm{FVS}(A)$ as vector space, we have that $\psi = \phi_*$ thus ϕ_* is unique. We conclude that $(\mathrm{FVS}(A), \iota)$ is the free vector space over A. □

Remark 3.9. We view the free vector space as a subspace of the free Riesz space.

Theorem 3.10. *Let B be a set and let $A \subset B$ be a subset. Let $(FRS(B), \iota_B)$ be the free Riesz space over B. Let $FRS(A)$ be the Riesz subspace of $FRS(B)$ generated by the elements $\iota_B(a)$, where $a \in A$, and let $\iota_A = \iota_B|_A$. Then $(FRS(A), \iota_A)$ is the free Riesz space over A.*

Proof. Let B be a set and let $A \subset B$ be a subset. The map $j_A : \mathbb{R}^{\mathbb{R}^A} \to \mathbb{R}^{\mathbb{R}^B}$ is an injective Riesz homomorphism, and hence $j_A|_{\mathrm{FRS}(A)} \to \mathrm{FRS}(B)$ is an injective Riesz homomorphism. □

Proposition 3.11. *Let A be a finite set. Let $(FRS(A), \iota)$ be the free Riesz space over A. Then $\sum_{a \in A} |\iota(a)|$ is a strong order unit for $FRS(A)$.*

Proof. Let A be a finite set with free Riesz space $(\mathrm{FRS}(A), \iota)$. Then the statement is clear from the fact that A is finite and that $\mathrm{FRS}(A)$ is generated by elements $\iota(a)$. □

Proposition 3.12. *Let A be a non-empty set and let $\mathcal{F}(A)$ denote the set of all finite subsets of A. Then*

$$FRS(A) = \bigcup_{B \in \mathcal{F}(A)} FRS(B),$$

where we view $FRS(B)$ as a Riesz subspace of $FRS(A)$.

Proof. Let A be a non-empty set with free Riesz space $(\mathrm{FRS}(A), \iota)$. From Theorem 3.10 it is clear that

$$\bigcup_{B \in \mathcal{F}(A)} \mathrm{FRS}(B) \subset \mathrm{FRS}(A).$$

On the other hand every element $x \in \mathrm{FRS}(A)$ is generated by finitely many elements $\iota_{a_1}, \dots, \iota_{a_n}$, so $x \in \mathrm{FRS}(\{a_1, \dots, a_n\})$ and hence

$$\mathrm{FRS}(A) = \bigcup_{B \in \mathcal{F}(A)} \mathrm{FRS}(B).$$

□

Proposition 3.13. *Let A be a set.*

1. *If $\xi \in \mathbb{R}^A$, then $\omega_\xi : FRS(A) \to \mathbb{R}$ defined by $\omega_\xi(f) = f(\xi)$, where $f \in FRS(A)$, is a Riesz homomorphism.*

2. *If $\omega : FRS(A) \to \mathbb{R}$ is a Riesz homomorphism, then there is a $\xi \in \mathbb{R}^A$ such that $\omega = \omega_\xi$.*

Here, we view $FRS(A)$ as a Riesz subspace of $\mathbb{R}^{\mathbb{R}^A}$.

Proof. The first statement is trivial. Suppose $\omega : \mathrm{FRS}(A) \to \mathbb{R}$ is a Riesz homomorphism. Define $\xi \in \mathbb{R}^A$ by $\xi(a) = \omega(\iota(a))$, for $a \in A$. Then for $a \in A$ we have $\omega(\iota(a)) = \xi(a) = \iota(a)(\xi) = \omega_\xi(\iota(a))$. Thus ω and ω_ξ coincide on the set $\{\iota(a) : a \in A\}$ that generates $\mathrm{FRS}(A)$ as Riesz space hence $\omega = \omega_\xi$. □

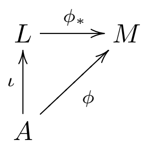
3.2 Free normed Riesz space and free Banach lattice

Something similar to free Riesz space exists for Banach lattices. It is easy to generalize the results of De Pagter and Wickstead for Banach lattices [14] to normed Riesz spaces. We give an overview of the main results.

Definition 3.14. Let A be a non-empty set and let X be a normed space. A map $\phi : A \to X$ is (norm) bounded, if there exists an $M > 0$ such that $||\phi(a)|| \leq M$, for all $a \in A$. We define the norm of a (norm) bounded map $\phi : A \to X$ to be $||\phi|| = \sup\{||\phi(a)|| : a \in A\}$.

Remark 3.15. Note that $||\cdot||$ is norm on the vector space of all norm bounded maps $\phi : A \to X$.

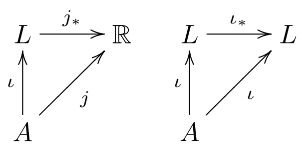
Definition 3.16. Let A be a non-empty set. The free normed Riesz space (free Banach lattice) over A is a pair (L, ι) where L is a normed Riesz space (Banach lattice) and $\iota : A \to L$ is a bounded map such that for any normed Riesz space (Banach lattice) M and for any bounded map $\phi : A \to M$ there is a Riesz homomorphism $\phi_* : L \to M$ with the property $||\phi_*|| = ||\phi||$. Moreover ϕ_* is the unique Riesz homomorphism $\psi : L \to M$ that satisifies $\phi = \psi \circ \iota$.



From the fact that $\iota_* = id_L$ follows that $||\iota|| = ||\iota_*|| = 1$. But even more is true.

Lemma 3.17. *([14, Proposition 4.2].) Let A be a non-empty set, and suppose that (L, ι) is a free Banach lattice or free normed Riesz space over A. Then $||\iota(a)|| = 1$, for every $a \in A$.*

Proof. Let A be a non-empty set, and suppose that (L, ι) is a free Banach lattice over A. Define $j : A \to \mathbb{R}$ by $a \mapsto 1, a \in A$. Then $||j_*|| = ||j|| = 1$. Further, $1 = ||j(a)|| = ||j_*(\iota(a))|| \leq ||j_*||\,||\iota(a)|| = ||\iota(a)||$, thus $||\iota(a)|| \geq 1$, for all $a \in A$. Since $||\iota|| = 1$ we have $||\iota(a)|| \leq 1$, for all $a \in A$. Therefore $||\iota(a)|| = 1$, for all $a \in A$.
The case that (L, ι) is a free normed Riesz space over A is proven similarly.



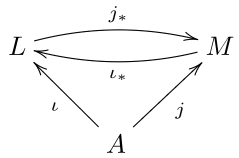
□

The free Banach lattice or normed Riesz space is unique if it exists, see [14, Proposition 4.3].

Lemma 3.18. *Let A be a non-empty set, and suppose that (L, ι) and (M, j) are free Banach lattices over A (free normed Riesz spaces over A.) Then there exists a unique isometric order isomorphism $\phi : L \to M$, such that $\phi \circ \iota = j$.*

Proof. Let A be a non-empty set, and suppose that (L, ι) and (M, j) are free Banach lattices over A. Let $j_* : L \to M$ be the unique Riesz homomorphism such that $j = j_* \circ \iota$. Then by Lemma 3.17, $||j_*|| = ||j|| = 1$. Similarly, there is a unique Riesz homomorphism $\iota_* : M \to L$ of norm 1, such that $\iota = \iota_* \circ j$. So $\iota_* \circ j_* : L \to L$ is a Riesz homomorphism and $\iota = \iota_* \circ j = \iota_* \circ j_* \circ \iota$. From the uniqueness statement in the theorem follows that $\iota_* \circ j_*$ is the identity map on L. Similarly, $j_* \circ \iota_*$

is the identity map on M. So ι_* is an isometric order isomorphism.
The case that (L, ι) is a free normed Riesz space over A is proven similarly.



□

For an ordered vector space E we denote by $E^\sim$ the space of all order bounded linear functionals on E. According to [1, Theorem 1.18] $E^\sim$ is equal to the space of regular linear functionals as soon as E is an Archimedean Riesz space. De Pagter and Wickstead define in [14] a lattice norm $||\cdot||_F$ on $\mathrm{FRS}(A)$, that turns $\mathrm{FRS}(A)$ into the free normed Riesz space over A and its norm completion is the free Banach lattice over A. We will now review this construction.

Definition 3.19. ([14, Definition 4.4]) For a non-empty set A we define a map $||\cdot||^\dagger : \mathrm{FRS}(A)^\sim \to [0,\infty]$ by

$$||\phi||^\dagger = \sup\{|\phi|(|\iota(a)|) : a \in A\}.$$

Let

$$\mathrm{FRS}(A)^\dagger = \{\phi \in \mathrm{FRS}(A)^\sim : ||\phi||^\dagger < \infty\}.$$

The following is clear from the definition.

Lemma 3.20. *$FRS(A)^\dagger$ is a vector lattice ideal in $FRS(A)^\sim$.*

Lemma 3.21. *Let A be a non-empty set. Let $\xi \in \mathbb{R}^A$. Let $\omega_\xi : FRS(A) \to \mathbb{R}$ be defined through $\omega_\xi(x) = x(\xi), x \in FRS(A)$ (See Proposition 3.13 on page 7). Then $||\omega_\xi|| < \infty$ if and only if ξ is bounded. Or, equivalently, $\omega_\xi \in FRS(A)^\dagger$ if and only if ξ is bounded.*

Proof. Let $\xi \in \mathbb{R}^A$. Since $|\xi(a)| = |\iota(a)(\xi)| = |\omega_\xi(\iota(a))| = \omega_\xi(|\iota(a)|) = |\omega_\xi|(|\iota(a)|)$, for all $a \in A$, it follows that $||\omega_\xi||^\dagger < \infty$ if and only if ξ is bounded. □

Lemma 3.22. *If A is a non-empty set, then $||\cdot||^\dagger$ is a Riesz norm on $FRS(A)^\dagger$.*

Proof. Let A be a non-empty set. $||\cdot||^\dagger$ is clearly a Riesz seminorm. Suppose that $||\phi||^\dagger = 0$. Then $|\phi|(|\iota(a)|) = 0$, for all $a \in A$, thus $\phi(|\iota(a)|) = 0$, for all $a \in A$. Let $x \in \mathrm{FRS}(A)$. By Proposition 3.12 on page 7 there are finitely many $a_1, \dots, a_n \in A$ such that $x \in \mathrm{FRS}(\{a_1, \dots, a_n\})$. By Proposition 3.11 on page 7 $e = \sum_{i=1}^n |\iota(a)|$ is a strong order unit for $\mathrm{FRS}(\{a_1, \dots, a_n\})$. Thus there is an $\lambda \in \mathbb{R}^+$ such that $|x| \le \lambda e$. It follows that $|\phi(x)| \le |\phi|(|x|) \le \lambda|\phi|(e) = 0$ and hence $\phi(x) = 0$. So $\phi = 0$. We conclude that $||\cdot||^\dagger$ is a Riesz norm. □

Definition 3.23. For a non-empty set A and for $x \in \mathrm{FRS}(A)$ we define $||x||_F = \sup\{\phi(|x|) : \phi \in (\mathrm{FRS}(A)^\dagger)^+, ||\phi||^\dagger \le 1\}$.

Theorem 3.24. *If A is a non-empty set, then $||\cdot||_F$ is a lattice norm on $FRS(A)$.*

Proof. From the definition it is clear that for $x, y \in \mathrm{FRS}(A)$ with $|x| \le |y|$ we have that $||x||_F \le ||y||_F$ and that $||\cdot||_F$ is positive homogenious and subadditive. First we will show that for all $x \in \mathrm{FRS}(A)$ we have that $||x||_F < \infty$. Let $x \in \mathrm{FRS}(A)$. By Proposition 3.12 on page 7 there are finite $a_1, \dots, a_n \in A$ such that $x \in \mathrm{FRS}(\{a_1, \dots, a_n\})$. By Proposition 3.11 on page 7 $e = \sum_{i=1}^n |\iota(a_i)|$ is a strong order unit of $\mathrm{FRS}(\{a_1, \dots, a_n\})$. So there is a $\lambda > 0$ such that $|x| \le \lambda e$. So $||x||_F \le \lambda ||e||_F$. For $\phi \in (\mathrm{FRS}(A)^\dagger)^+, ||\phi||^\dagger \le 1$, we have $\phi(e) = \sum_{i=1}^n \phi(|\iota(a_i)|) \le n$. Thus $||e||_F \le n$ and hence $||x||_F \le \lambda n < \infty$.
It only remains to show that for all $x \in \mathrm{FRS}(A)$ we have that $||x||_F = 0$ implies that $x = 0$. Let $x \in \mathrm{FRS}(A)$ with $||x||_F = 0$. Clearly, for all $\phi \in (\mathrm{FRS}(A)^\dagger)^+$ we have $\phi(x) = 0$. In particular

$\omega_\xi(x) = x(\xi) = 0$, for every $\xi \in \mathbb{R}^A$. By Proposition 3.12 on page 7 there is a finite subset B of A such that $x \in \mathrm{FRS}(B)$. In particular for every $\xi \in \mathbb{R}^B \subset \mathbb{R}^A$ we have that $x(\xi) = 0$. But that means that $x = 0$, since $x \in \mathrm{FRS}(B) \subset \mathbb{R}^{\mathbb{R}^B}$. □

Theorem 3.25. *Let A be a non-empty set. Then $((FRS(A), ||\cdot||_F), \iota)$ is the free normed Riesz space over A.*

Proof. Let A be a non-empty set. Consider the free Riesz space over $A, (\mathrm{FRS}(A), \iota)$. Let M be an arbitrary normed Riesz space and let $\phi : A \to M$ a bounded map that maps into M_1, the closed unit ball of M, and suppose that $||\phi|| = 1$. By the definition of the free Riesz space, there is a unique Riesz homomorphism $\phi_* : \mathrm{FRS}(A) \to M$ such that $\phi = \phi_* \circ \iota$. It only remains to prove that $||\phi_*|| = ||\phi||$.
By Lemma 3.17 on page 8 we have for all $a \in A, ||\iota(a)|| = 1$. So $||\phi_*|| \geq ||\phi_*(\iota(a))|| = ||\phi(a)||$, thus $||\phi_*|| \geq ||\phi||$. Suppose that $||\phi_*|| > ||\phi||$. So for some $x \in \mathrm{FRS}(A)$ with $||x|| = 1$ is $||\phi_*(x)|| = ||\phi_*(|x|)|| > ||\phi||$. By Hahn-Banach there exists a positive functional ψ on M of norm at most one and $\psi(\phi_*|x|) > ||\phi||$.
We have $||\psi \circ \phi_*||^\dagger = \sup\{|\psi \circ \phi_*|(|\iota(a)|) : a \in A\} = \sup\{|\psi \circ \phi_*(\iota(a))| : a \in A\} = \sup\{|\psi(\phi(a))| : a \in A\} \leq 1$. Thus $||x||_F \geq ||\psi(\phi_*(|x|))|| > \phi = 1$. This is a contradiction. Therefore $||\phi_*|| = ||\phi||$.
Suppose $\phi : \mathrm{FRS}(A) \to M$ is a bounded map and $||\phi|| > 0$. Then $\bar{\phi} = \phi/||\phi||$ is of norm one and maps into M_1. It is clear that $\bar{\phi}_* = \phi_*/||\phi||$, so $||\phi_*|| = ||\phi||$. Suppose $\phi : \mathrm{FRS}(A) \to M$ is zero. Then $\phi_* = 0$, so $||\phi_*|| = ||\phi||$. This concludes our proof. □

Remark 3.26. We denote the free normed Riesz space over a non-empty set A by $\mathrm{FNRS}(A)$.

For the next theorem see [1, Theorems 4.1 and 4.2].

Theorem 3.27. *Let E be a normed Riesz space and let E' denote its norm dual space. Then E' is a Banach lattice. Consider E as a subspace of its double dual $E'' = (E')'$. Then the norm completion of E is a Banach lattice.*

According to [1, Theorem 4.3] we have.

Theorem 3.28. *Let L be a Banach lattice and M a normed Riesz space. Every positive linear map $\phi : L \to M$ is continuous.*

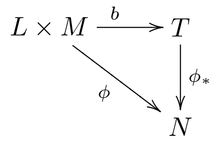
Theorem 3.29. *Let A be a non-empty set. Let $FBL(A)$ be the $||\cdot||_F$-norm completion of $FRS(A)$. Then $FBL(A)$ is a Banach lattice and $(FBL(A), \iota)$ is the free Banach lattice over A.*

Proof. Let A be a non-empty set and let $\mathrm{FBL}(A)$ be the $||\cdot||_F$-norm completion of $\mathrm{FRS}(A)$. By Theorem 3.27, $\mathrm{FBL}(A)$ is a Banach lattice. Suppose M is Banach lattice and $\phi : A \to M$ a bounded map. By Theorem 3.25 there is a Riesz homomorphism $\phi_* : \mathrm{FRS}(A) \to M$ such that $\phi = \phi_* \circ \iota$ and $||\phi_*|| = ||\phi||$, moreover ϕ_* is the unique Riesz homomorphism ψ that satisfies $\phi = \psi \circ \iota$. The Riesz homomorphism ϕ_* extends by continuity to a Riesz homomorphism $\phi_* : \mathrm{FBL}(A) \to M$ and we still have $\phi = \phi_* \circ \iota$ and $||\phi_*|| = ||\phi||$. Suppose $\psi : \mathrm{FBL}(A) \to M$ is also a Riesz homomorphism that satisfies $\phi = \psi \circ \iota$. Since $\mathrm{FBL}(A)$ and M are Banach lattices, by Theorem 3.28 ψ is continuous. Note that ψ and ϕ_* coincide on $\mathrm{FRS}(A)$. Since the continuous extension is unique, we have that $\psi = \phi_*$. So $(\mathrm{FBL}(A), \iota)$ is the free Banach lattice over A. □

4 The Archimedean Riesz tensor product

Here we study the construction of the tensor product of (Archimedean) Riesz spaces due to B. de Pagter.
The following definition is taken from [13].

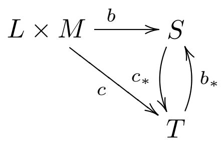
Definition 4.1. Let L and M be (Archimedean) Riesz spaces. The (Archimedean) Riesz tensor product of L and M is a pair (T, b) where T is an (Archimedean) Riesz space and $b : L \times M \to T$ a Riesz bimorphism, such that for every (Archimedean) Riesz space N and every Riesz bimorphism $\phi : L \times M \to N$ there is a unique Riesz homomorphism $\phi_* : T \to N$ such that $\phi = \phi_* \circ b$.



The next theorem can be found in [13]

Theorem 4.2. *The (Archimedean) Riesz tensor product is unique if it exists, in the following sense: let L and M be (Archimedean) Riesz spaces and suppose that (S, b) and (T, c) are two (Archimedean) Riesz tensor products of L and M, then there is a unique bijective linear map $\phi : S \to T$ such that ϕ and ϕ^{-1} are Riesz homomorphisms and $\phi \circ b = c$. In particular, ϕ is an order isomorphism.*

Proof. Let L and M be (Archimedean) Riesz spaces and suppose that (S, b) and (T, c) are two (Archimedean) Riesz tensor products of L and M. By definition there is unique Riesz homomorphism $c_* : S \to T$ such that $c = c_* \circ b$ and a unique Riesz homomorphism $b_* : T \to S$ such that $b = b_* \circ c$. So $b = b_* \circ c = b_* \circ c_* \circ b$. Note that $b_* \circ c_* : S \to S$ is a Riesz homomorphism and that also the identity map id_S on S is a Riesz homomorphism such that $b = id_S \circ b$. From the uniqueness statement it follows that $b_* \circ c_* = id_S$. Likewise we have that $c_* \circ b_*$ is the identity map on T. Define $\phi = c_* : S \to T$. Then ϕ is bijective, ϕ and $\phi^{-1} = b_* : T \to S$ are Riesz homomorphisms, $c = \phi \circ b$ and ϕ is the unique Riesz homomorphism with these properties. In particular ϕ is an order isomorphism.



□

Theorem 4.3. *Let L and M be (Archimedean) Riesz spaces. If (T, b) is the (Archimedean) Riesz tensor product of L and M, then T is generated as Riesz space by elements $b(x, y), x \in L, y \in M$.*

Proof. Let L, M and (T, b) be as in the theorem. Let S be the Riesz subspace of T generated by the elements $b(x, y), x \in L, y \in M$. Let N be an arbitrary (Archimedean) Riesz space and $\phi : L \times M \to N$ a Riesz bimorphism. Let $\phi'_* : T \to N$ be the unique Riesz homomorphism with $\phi = \phi'_* \circ b$. Note that b maps into S. Let $\phi_* : S \to N$ be the restriction of ϕ'_* to S. Then ϕ_* is a Riesz homomorphism and $\phi = \phi_* \circ b$. Let $\psi : S \to N$ be any Riesz homomorphism with $\phi = \psi \circ b$. For all $x \in L$ and $y \in M$ we have that $\phi_*(b(x, y)) = \psi(b(x, y))$. Since S is generated as Riesz space by elements $b(x, y), x \in L, y \in M$, we have that $\phi_* = \psi$. Thus (S, b) is also the (Archimedean) Riesz tensor product of L and M. From Theorem 4.2 follows that $S = T$. This concludes our proof. □

With thanks to B. de Pagter [13] we have the following nice representation of the (Archimedean) Riesz tensor product.

Theorem 4.4. *Let L and M be (Archimedean) Riesz spaces. Let J be the ideal in $FRS(L \times M)$ generated by the elements*

$$\begin{cases} \iota(\alpha x + \beta y, z) - \alpha\iota(x, z) - \beta\iota(y, z), & x, y \in L, z \in M, \alpha, \beta \in \mathbb{R}, \\ \iota(x, \alpha y + \beta z) - \alpha\iota(x, y) - \beta\iota(x, z), & x \in L, y, z \in M, \alpha, \beta \in \mathbb{R}, \\ |\iota(x, y)| - \iota(|x|, |y|), & x \in L, y \in M. \end{cases}$$

Let $J_1 = J$, and in the Archimedean case, let J_1 be the uniform closure of J in $FRS(L \times M)$. Let $T := FRS(L \times M)/J_1$ and let $q : FRS(L \times M) \to T$ be the quotient map. Let $b : L \times M \to T$ be defined by $b(x, y) = q(\iota(x, y))$. Then b is a Riesz bimorphism and (T, b) is the (Archimedean) Riesz tensor product of L and M.

Proof. Let L and M be (Archimedean) Riesz spaces and let J, J_1, T, q and b be as in the theorem.

Claim 4.5. *b is a Riesz bimorphism.*

Proof of Claim 4.5. Note that for all $x, y \in L$ and $z \in M$ and for all $\alpha, \beta \in \mathbb{R}$, we have that

$$\begin{aligned} b(\alpha x + \beta y, z) &= q(\iota(\alpha x + \beta y, z)) \\ &= q(\alpha\iota(x, z) + \beta\iota(y, z)) \\ &= \alpha q(\iota(x, z)) + \beta q(\iota(y, z)) \\ &= \alpha b(x, z) + \beta b(y, z). \end{aligned}$$

Likewise, for all $x \in L$ and $y, z \in M$ and for all $\alpha, \beta \in \mathbb{R}$ we have that $b(x, \alpha y + \beta z) = \alpha b(x, y) + \beta b(y, z)$. Thus b is bilinear.
Note that for all $x \in L$ and $y \in M$ we have

$$\begin{aligned} b(|x|, |y|) &= q(\iota(|x|, |y|)) \\ &= q(|\iota(x, y)|) \\ &= |q(\iota(x, y))| \\ &= |b(x, y)| \end{aligned}$$

Hence b is a Riesz bimorphism. □

Let N be an arbitrary (Archimedean) Riesz space. Let $\phi : L \times M \to N$ be a Riesz bimorphism. By definition, there is a unique Riesz homomorphism $\psi : \mathrm{FRS}(L \times M) \to N$ such that $\phi = \psi \circ \iota$.

Claim 4.6. $J_1 \subset \ker \psi$.

Proof of Claim 4.6. Since ϕ is a Riesz homomorphism, we have that $\phi(\alpha x + \beta y, z) - \alpha\phi(x, z) - \beta\phi(y, z) = 0$ for all $x, y \in L, z \in M$ and $\alpha, \beta \in \mathbb{R}$. Thus

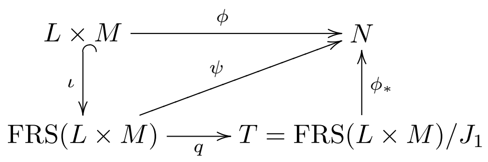
$$\begin{aligned} 0 &= \phi(\alpha x + \beta y, z) - \alpha\phi(x, z) - \beta\phi(y, z) \\ &= \psi(\iota(\alpha x + \beta y, z)) - \alpha\psi(\iota(x, z)) - \beta\psi(\iota(y, z)) \\ &= \psi(\iota(\alpha x + \beta y, z) - \alpha\iota(x, z) - \beta\iota(y, z)) \end{aligned}$$

So $\iota(\alpha x + \beta y, z) - \alpha\iota(x, z) - \beta\iota(y, z) \in \ker \psi$. Likewise, for all $x \in L$ and for all $y, z \in M$ and for all $\alpha, \beta \in \mathbb{R}$ we have that $\iota(x, \alpha y + \beta z) - \alpha\iota(x, y) - \beta\iota(x, z) \in \ker \psi$. Since ϕ is a Riesz bimorphism, we have

$$\begin{aligned} 0 &= |\phi(x, y)| - \phi(|x|, |y|) \\ &= |\psi(\iota(x, y))| - \psi(\iota(|x|, |y|)) \\ &= \psi(|\iota(x, y)|) - \psi(\iota(|x|, |y|)) \\ &= \psi(|\iota(x, y)| - \iota(|x|, |y|)), \end{aligned}$$

for all $x \in L$ and $y \in M$. Thus $|\iota(x, y)| - \iota(|x|, |y|) \in \ker \psi$. We have that $\ker \psi$ is an ideal and it contains the set that generates $J = J_1$, so $J_1 \subset \ker \psi$, and in the Archimedean case by Proposition 2.12 on page 4, we have that $\ker \psi$ is relatively uniformly closed too, thus $J_1 \subset \ker \psi$. □

Define $\phi_* : T \to N$ by $\phi_*(q(x)) = \psi(x), x \in \mathrm{FRS}(L \times M)$. Suppose that $q(x) = q(y)$, for $x, y \in \mathrm{FRS}(L \times M)$, then $x - y \in J_1 \subset \ker \psi$, so $\psi(x) = \psi(q)$. Thus ϕ_* is well defined and clearly a Riesz homomorphism.
Suppose $\chi : T \to N$ is a Riesz homomorphism that satisfies $\phi = \chi \circ b$. Then, for all $x \in L, y \in M$, we have $\chi(b(x, y)) = \chi(q(\iota(x, y)) = \phi(x, y) = \phi_*(q(\iota(x, y)) = \psi(\iota(x, y)))$. By Theorem 3.5 on page 6, $\mathrm{FRS}(L \times M)$ is generated as Riesz space, by $\{\iota(x, y) : x \in L, y \in M\}$ and $\chi \circ q$ is a Riesz homomorphism. Thus we have that $\chi \circ q = \psi$. Thus $\chi = \phi_*$. It follows that (T, b) is the (Archimedean) Riesz tensor product of L and M.



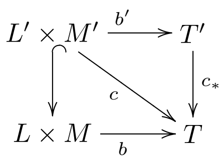
□

Remark 4.7.

1. If L and M are Archimedean Riesz spaces, then both the Riesz tensor product and the Archimedean Riesz tensor product exists, but are in general not isomorphic, see [13].
2. For (Archimedean) Riesz spaces L and M we denote the Riesz tensor product by $L\tilde{\otimes}M$ and the Archimedean Riesz tensor product by $L\bar{\otimes}M$. The constructed Riesz bimorphism b in the previous theorem is denoted by $\otimes$. For $x \in L$ and $y \in M$ we define $x \otimes y = b(x, y)$.
3. From the construction follows that $L\tilde{\otimes}M$ is Riesz isomorphic to $M\tilde{\otimes}L$ and that $L\bar{\otimes}M$ is Riesz isomorphic to $M\bar{\otimes}L$.

Theorem 4.8. *Suppose L and M are Riesz spaces, L' is a Riesz subspace of L and M' is a Riesz subspace of M. Let (T, b) be the (Archimedean) Riesz tensor product of L and M. Let S be the Riesz subspace of T generated by elements $b(x, y), x \in L', y \in M'$. Let $c = b|_{L' \times M'}$. Then (S, c) is the (Archimedean) Riesz tensor product of L' and M'.*

Proof. Let L and M be (Archimedean) Riesz spaces. Let L' a Riesz subspace of L and M' a Riesz subspace of M. Let (T', b') the (Archimedean) Riesz tensor product of L' and M' and let (T, b) be the (Archimedean) Riesz tensor product of L and M. Note that $c = b|_{L' \times M'} : L' \times M' \to T$ is a Riesz bimorphism. By definition there is a Riesz homomorphism $c_* : T' \to T$ such that $c_* \circ b' = c$. Thus $b(x, y) = c_*(b'(x, y))$, for all $x \in L', y \in M'$. Thus $c_* : T' \to T$ is injective. It follows that we can view T' via c_* as a Riesz subspace of T generated by elements $b(x, y), x \in L', y \in M'$.



□

5 The positive tensor product

In this section we define the positive tensor product, prove some elementary properties and give two constructions of the positive tensor product. One construction is due to Van Gaans and Kalauch and the other construction is new.
Positive bimorphisms are just what one thinks they are.

Definition 5.1. Let E, F and G be partially ordered vector spaces and let $b : E \times F \to G$ be bilinear. Then we call b a positive bimorphism or positive bilinear map, if $b(x, \cdot)$ is positive, for all $x \in E^+$, and $b(\cdot, y)$ is positive, for all $y \in F^+$. This is equivalent to $b(x, y) \geq 0$ for all $x \in E^+$ and $y \in F^+$.

Definition 5.2. Let E and F be integrally closed pre-Riesz spaces. A pair (T, b), where T is an integrally closed pre-Riesz space and $b : E \times F \to T$ is a positive bimorphism, is a positive tensor product of E and F if for all integrally closed pre-Riesz spaces G and positive bimorphisms $\phi : E \times F \to G$ there is a unique positive linear map $\phi_* : T \to G$ such that $\phi = \phi_* \circ b$.

$$\begin{array}{ccc} E \times F & \xrightarrow{b} & T \\ & \searrow^{\phi} & \downarrow{\scriptstyle \phi_*} \\ & & G \end{array} \tag{1}$$

The positive tensor product is unique, if it exists. In the following sense.

Theorem 5.3. *Let E and F be integrally closed pre-Riesz spaces and suppose that (S, b) and (T, c) are positive tensor products of E and F. Then there is a unique order isomorphism $\phi : S \to T$ such that $\phi \circ b = c$.*

Proof. The proof is similar to the proof of Theorem 4.2 on page 11. □

Theorem 5.4. *Let E and F be integrally closed pre-Riesz spaces. If (T, b) is the positive tensor product of E and F, then T is generated as vector space by elements $b(x, y), x \in E, y \in F$.*

Proof. Let E, F and (T, b) as in the theorem. Let S be the subspace of T generated by the elements $b(x, y), x \in E, y \in F$. Clearly, S is integrally closed and generated by positive elements $b(x, y), x \in E^+, y \in F^+$ and hence directed. Thus S is an integrally closed pre-Riesz space. Let G be an arbitrary integrally closed pre-Riesz space and $\phi : E \times F \to G$ a positive bimorphism. Let $\phi'_* : T \to G$ be the unique positive linear map with $\phi = \phi'_* \circ b$. Note that b maps into S. Let $\phi_* : S \to G$ be the restriction of ϕ'_* to S. Then ϕ_* is a positive linear map and $\phi = \phi_* \circ b$. Let $\psi : S \to G$ be any positive linear map with $\phi = \psi \circ b$. For all $x \in E$ and $y \in F$ we have that $\phi_*(b(x, y)) = \psi(b(x, y))$. Since S is generated as vector space by elements $b(x, y), x \in E, y \in F$, we have that $\phi_* = \psi$. Thus (S, b) is also the positive tensor product of L and M. From Theorem 5.3 follows that $S = T$. This concludes our proof. □

Example 5.5. Let E be an integrally closed pre-Riesz space. We calculate the positive tensor product of $\mathbb{R}$ and E. Define $b : \mathbb{R} \times E \to E$ through $b(r, x) = rx$. Let F be an arbitrary integrally closed pre-Riesz space and let $\phi : \mathbb{R} \times E \to F$ be a positive bimorphism. Note that $\phi(r, x) = \phi(1, rx)$, for all $r \in \mathbb{R}$ and $x \in E$. Define $\phi_* : E \to F$ through $\phi_*(x) = \phi(1, x)$. Then ϕ_* is a positive linear map and for all $r \in \mathbb{R}$ and $x \in E$ we have $\phi(r, x) = \phi(1, rx) = \phi_*(rx) = \phi_*(b(r, x))$. Thus $\phi = \phi_* \circ b$. Let $\psi : E \to F$ be any positive linear map with $\phi = \psi \circ b$, then $\psi(x) = \psi(b(1, x)) = \phi(1, x) = \phi_*(x)$, for all $x \in E$. Thus $\phi_* = \psi$. It follows that (E, b) is the positive tensor product of $\mathbb{R}$ and E.

5.1 Tensor cones

In this section, we give a short overview of the results of Van Gaans and Kalauch [7]. We skip most of the proofs.

Definition 5.6. Let E and F be partially ordered vector spaces and let $E \otimes F$ be the usual vector space tensor product of E and F. We define the projective tensor cone K_T to be

$$K_T = \left\{ \sum_{i=1}^{n} \lambda_i x_i \otimes y_i : n \in \mathbb{N} \text{ and } \lambda_i \in \mathbb{R}^+, x_i \in E^+, y_i \in F^+, \text{ for } i \in \{1, \ldots, n\} \right\}.$$

It is just the wedge generated by positive elements $x \otimes y$. But it is in fact a cone and generates $E \otimes F$ according to the following lemma.

Lemma 5.7. *(See [7, Theorem 2.5].) Let E and F be partially ordered vector spaces, then the projective cone K_T is a cone and $(E \otimes F, K_T)$ is directed.*

Definition 5.8. (See [7, Definition 4.1].) Let E and F be integrally closed pre-Riesz spaces. A cone K_I in $E \otimes F$ is called an *integrally closed tensor cone* if the projective cone K_T is contained in K_I, $(E \otimes F, K_I)$ is an integrally closed pre-Riesz space and the following universal mapping is satisfied. For every integrally closed pre-Riesz space G and for every positive bilinear map $\phi : E \times F \to G$ the unique linear map $\phi_* : (E \otimes F, K) \to G$, with $\phi = \phi_* \circ \otimes$, is positive.

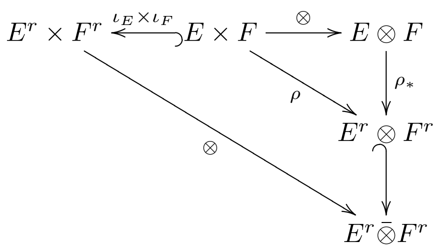
Remark 5.9. Van Gaans and Kalauch call the integrally closed tensor cone the Archimedean tensor cone.

Theorem 5.10. *Let E and F be integrally closed pre-Riesz spaces. Suppose K_1 and K_2 are integrally closed tensor cones of $E \otimes F$, then $K_1 = K_2$.*

Proof. Let E and F be integrally closed pre-Riesz spaces. Suppose K_1 and K_2 are integrally closed tensor cones of $E \otimes F$. Note that $\phi : E \times F \to E \otimes F$ defined by $\phi(x, y) = x \otimes y$ is positive with cones K_1 and K_2, since both contain K_T.
By definition there is a unique positive linear map $\phi_1 : (E \otimes F, K_1) \to (E \otimes F, K_2)$ such that $\phi_1(x \otimes y) = \phi(x, y) = x \otimes y$, for all $x \in E, y \in F$, and a unique positive linear map $\phi_2 : (E \otimes F, K_2) \to (E \otimes F, K_1)$ such that $\phi_2(x \otimes y) = \phi(x, y) = x \otimes y$, for all $x \in E, y \in F$. Thus $\phi_1 = \phi_2$ is the identity map on $E \otimes F$, and the identity map is bipositive hence $K_1 = K_2$. □

Definition 5.11. Let E and F be integrally closed pre-Riesz spaces. Let (E^r, ι_E) and (F^r, ι_F) be the Riesz completions of E and F respectively. Consider the Archimedean Riesz tensor product $E^r \bar{\otimes} F^r$ of E^r and F^r. We can view the usual vector space tensor product $E^r \otimes F^r$ as a subspace of $E^r \bar{\otimes} F^r$ with the induced ordering. Define $\rho : E \times F \to E^r \otimes F^r$ by $\rho(x, y) = (\otimes \circ (\iota_E, \iota_F))(x, y) = \iota_E(x) \otimes \iota_F(y)$. Then ρ is positive bilinear and induces a unique linear map $\rho_* : E \otimes F \to E^r \otimes F^r$ with $\rho(x, y) = \rho_*(x \otimes y)$ for all $x \in E$ and $y \in F$. By [7, Lemma 2.4] ρ_* is injective. We define the *Fremlin tensor cone* to be

$$K_F = \{x \in E \otimes F : \rho_*(x) \geq 0\}.$$



Proposition 5.12. *(See [7, Lemmas 4.2 and 4.3].) The Fremlin tensor cone is a generating integrally closed cone. Moreover $K_T \subset K_F$ and K_F is relatively uniformly closed in $(E \otimes F, K_T)$.*

Theorem 5.13. *(See [7, Theorem 4.4].) Let E and F be integrally closed pre-Riesz spaces. For a cone K in $E \otimes F$ the following four statements are equivalent.*

(i) K is the integrally closed tensor cone.

(ii) For all integrally closed pre-Riesz spaces (S, K_S) and for any linear map $\phi : E \otimes F \to S$ with $\phi(x) \in K_S$ for all $x \in K_T$, we also have that $\phi(x) \in K_S$ for all $x \in K$.

(iii) K is the intersection of all integrally closed cones in $E \otimes F$ that contain K_T.

(iv) K is the relatively uniformly closure of K_T in $(E \otimes F, K_T)$.

Proposition 5.12 and Theorem 5.13 yield the following.

Theorem 5.14. *For any pair of integrally closed pre-Riesz spaces E and F, the relatively uniformly closure K_I of K_T in $(E \otimes F, K_T)$ is a cone. That is, K_I is the integrally closed tensor cone in $E \otimes F$ and $(E \otimes F, K_I)$ is the positive tensor product of E and F.*

Proof. Let E and F be integrally closed pre-Riesz spaces and let K_I be the relatively uniformly closure of K_T in $(E \otimes F, K_T)$. By Proposition 5.12 on the preceding page we have that $K_T \subset K_F$ and that K_F is a relatively uniformly closed cone in $(E \otimes F, K_T)$. Since K_I is the relatively uniformly closure of K_T in $(E \otimes F, K_T)$, it follows that $K_I \subset K_F$. Thus K_I is a cone. From Theorem 5.13 on the previous page follows that K_I is the integrally closed tensor cone in $E \otimes F$ thus $(E \otimes F, K_I)$ is the positive tensor product of E and F. □

Remark 5.15. We denote the positive tensor product by $E \otimes F$.

From the construction follows clearly the following two theorems.

Theorem 5.16. *Let E', E, F' and F be integrally closed pre-Riesz spaces, such that $E' \subset E, F' \subset F$ then there exists a bipositive linear map $\phi : E' \otimes F' \to E \otimes F$ with $\phi(x \otimes y) = x \otimes y$.*

Theorem 5.17. *Let E and F be integrally closed pre-Riesz spaces. Then there is an order isomorphism $\phi : E \otimes F \to F \otimes E, \phi(x \otimes y) = y \otimes x, x \in E, y \in F$.*

5.2 Construction of the positive tensor product via a free Riesz space

Now we give another construction of the positive tensor product of two arbitrary integrally closed pre-Riesz spaces. This construction does not make use of the results of Fremlin.
Let E and F be two arbitrary integrally closed pre-Riesz spaces. We construct the positive tensor product (T, b) of E and F out the free Riesz space $(\mathrm{FRS}(E \times F), \iota)$ over $E \times F$, by choosing a suitable ideal J in $\mathrm{FRS}(E \times F)$ and by choosing a vector subspace T of $T' = \mathrm{FRS}(E \times F)/J$ generated by elements $q(\iota(x, y)), x \in E, y \in F$, where q is the quotient map. The positive bimorphism b is chosen to be $q \circ \iota$ restricted to T.

Theorem 5.18. *Let E and F be integrally closed pre-Riesz spaces. Consider the free Riesz space over $E \times F$ (seen as a set), $(FRS(E \times F), \iota)$. Let J be the intersection of all relatively uniformly closed ideals that contain the following set.*

$$\begin{cases} \iota(\alpha x + \beta y, z) - \alpha\iota(x, z) - \beta\iota(y, z), & x, y \in E, z \in F, \alpha, \beta \in \mathbb{R}, \\ \iota(x, \alpha y + \beta z) - \alpha\iota(x, y) - \beta\iota(x, z), & x \in E, y, z \in F, \alpha, \beta \in \mathbb{R}, \\ \iota(x, y) - |\iota(x, y)| & x \in E^+, y \in F^+. \end{cases} \tag{2}$$

Let $T' = FRS(E \times F)/J$ and let $q : FRS(E \times F) \to T'$ be the quotient Riesz homomorphism. Define $b' : E \times F \to T'$ by $b'(x, y) = q(\iota(x, y)) = (q \circ \iota)(x, y)$. Let $T = Span\{b'(x, y) : x \in E, y \in F\}$ and let $b : E \times F \to T$ be the restriction of b'. Then T is an integrally closed pre-Riesz space, b is a positive bilinear map and (T, b) is the positive tensor product of E and F.

Proof. Let E, F, J, T', T, b', b and q be as in the theorem. Since $\mathrm{FRS}(E \times F)$ is a Riesz space, J is a relatively uniformly closed ideal, by Theorem 2.11 on page 4 we have that T' is an Archimedean Riesz space.

Claim 5.19. *b' is a positive bimorphism.*

Proof of Claim 5.19. Let $x, y \in E$ and $z \in F$. Let $\alpha, \beta \in \mathbb{R}$. Then $0 = q(\iota(\alpha x + \beta y, z) - \alpha\iota(x, z) - \beta\iota(y, z)) = q(\iota(\alpha x + \beta y, z)) - \alpha q(\iota(x, z)) - \beta q(\iota(y, z)) = b'(\alpha x + \beta y, z) - \alpha b'(x, z) - \beta b'(y, z)$. Therefore $b'(\alpha x + \beta y, z) = \alpha b'(x, z) + \beta b'(y, z)$. Likewise, we have for all $x \in E, y, z \in F$ and $\alpha, \beta \in \mathbb{R} : b'(x, \alpha y + \beta z) = \alpha b'(x, y) + \beta b'(x, z)$. Thus b' is bilinear. Let $x \in E^+, y \in F^+$. Then $b'(x, y) = q(\iota(x, y)) = q(|\iota(x, y)|) = |q(\iota(x, y))| = |b'(x, y)| \geq 0$. Hence b' is a positive bimorphism. □

Note that T is generated as subspace of T' by positive elements $b'(x, y), x \in E^+, y \in F^+$, thus T is a directed integrally closed partially ordered vector space and hence an integrally closed pre-Riesz space. Since $b'(E \times F) \subset T$, we have that b is well defined and clearly a positive bilinear map. Let G be an arbitrary integrally closed pre-Riesz space and let $\phi : E \times F \to G$ be an arbitrary positive bimorphism. Let G^r be the Riesz completion of G and we may assume that $G \subset G^r$. We view ϕ as a positive bimorphism from $E \times F$ to G^r. Let $\psi : \mathrm{FRS}(E \times F) \to G^r$ be the unique Riesz bimorphism that satisfies $\phi = \psi \circ \iota$.

$$\begin{array}{ccc} \mathrm{FRS}(E\times F) & \xrightarrow{\psi} & G^r \\ {\scriptstyle\iota}\uparrow & \nearrow{\scriptstyle\phi} & \\ E\times F & & \end{array} \tag{3}$$

Note that $\ker(\psi)$ is a relatively uniformly closed ideal of $\mathrm{FRS}(L\times M)$. We will show that $J \subset \ker(\psi)$. To do that, it is sufficient to show that the set that generates J is contained in $\ker(\psi)$. Note that for all $\alpha, \beta \in \mathbb{R}$ and $x, y \in E$ and $z \in F$ we have $\psi(\iota(\alpha x + \beta y, z) - \alpha\iota(x, z) - \beta\iota(y, z)) = \psi(\iota(\alpha x + \beta y, z)) - \alpha\psi(\iota(x, z)) - \beta\psi(\iota(y, z)) = \phi(\alpha x + \beta y, z) - \alpha\phi(x, z) - \beta\phi(y, z) = 0$. So $\iota(\alpha x + \beta y, z) - \alpha\iota(x, z) - \beta\iota(y, z) \in \ker(\psi)$. Likewise $\iota(x, \alpha y + \beta z) - \alpha\iota(x, y) - \beta\iota(x, z) \in \ker(\psi)$, for all $x \in E, y, z \in F$ and $\alpha, \beta \in \mathbb{R}$. Let $x \in E^+, y \in F^+$. Since $\phi(x, y) \geq 0$, we have that $\psi(\iota(x, y) - |\iota(x, y)|) = \psi(\iota(x, y)) - |\psi(\iota(x, y))| = \phi(x, y) - |\phi(x, y)| = 0$. Thus $\iota(x, y) - |\iota(x, y)| \in \ker(\psi)$ for all $x \in E^+, y \in F^+$. We conclude that $J \subset \ker(\psi)$. Now define $\phi'_* : T' \to G^r$ by $\phi'_*(q(x)) = \psi(x)$. Suppose $q(x) = q(y)$ for $x, y \in \mathrm{FRS}(E \times F)$. Then $y - x \in J \subset \ker(\psi)$. Thus $\phi_*(q(x)) = \psi(x) = \psi(x + y - x) = \psi(y) = \phi_*(q(y))$. It follows that ϕ'_* is well defined. Clearly, ϕ'_* is linear. Let $x \in \mathrm{FRS}(E \times F)$. Then $\phi'_*(|q(x)|) = \phi'_*(q(|x|)) = \psi(|x|) = |\psi(x)| = |\phi'_*(q(x))|$, since ψ is a Riesz homomorphism. Hence ϕ'_* is a Riesz homomorphism and in particular a positive linear map and $\phi'_*(b'(x, y)) = \phi'_*(q(\iota(x, y)) = \psi(\iota(x, y)) = \phi(x, y)$.
It is clear that $\phi = \phi_* \circ b$ and that b is positive. It remains to show that ϕ_* is the unique positive linear map with this property. Let $\chi : T \to G$ be a positive linear map that satisfies $\phi = \chi \circ b$. Let $t \in T$. Then there are $x_1, \ldots, x_n \in E, y_1, \ldots, y_n \in F$ such that $t = \sum_{i=1}^n b(x_i, y_i)$. So $\chi(t) = \sum_{i=1}^n \chi(b(x_i, y_i)) = \sum_{i=1}^n \phi(x_i, y_i) = \sum_{i=1}^n \phi_*(b(x_i, y_i)) = \phi_*(t)$. In fact, we proved that ϕ_* is the unique *linear map* χ with the property that $\phi = \chi \circ b$. Hence (T, b) is the positive tensor product of E and F. □

We have shown something stronger. Remark that we only use the fact that E and F are partially ordered vector spaces, that G is pre-Riesz, but not that it is integrally closed. Note that T is directed if E and F are directed. Therefore we have the following.

Theorem 5.20. *Let E and F be partially ordered vector spaces. Then there exists a pair (T, b) where T is an integrally closed partially ordered vector space and $b : E \times F \to T$ is a positive bimorphism, such that for any pre-Riesz space G (either integrally closed or not integrally closed) and for any positive bimorphism $\phi : E \times F \to G$ there exist a unique linear map $\phi_* : T \to G$, such that $\phi = \phi_* \circ b$. Moreover ϕ_* is positive. If E and F are directed, then T is an integrally closed pre-Riesz space.*

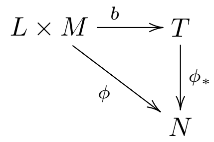
$$\begin{array}{ccc} E\times F & \xrightarrow{b} & T \\ & {\scriptstyle\phi}\searrow & \downarrow{\scriptstyle\phi_*} \\ & & G \end{array} \tag{4}$$

6 The normed Riesz space tensor product and Banach lattice tensor product

If Riesz spaces L and M have a Riesz norm, then it will be natural if the Archimedean Riesz tensor product also has a Riesz norm, likewise the tensor product of Banach lattices should be a Banach lattice. In this section we define and study the Banach lattice tensor product. We give a construction of the Banach lattice tensor product as a quotient of a free Banach lattice.

6.1 Definitions and properties

Definition 6.1. Let L and M be normed Riesz spaces (Banach lattices). The normed Riesz space (Banach lattice) tensor product of L and M is a pair (T, b), where T is a normed Riesz space (Banach lattice) and $b : L \times M \to T$ is a continuous Riesz bimorphism with $||b|| \leq 1$ and the property that for every normed Riesz space (Banach lattice) N and for every continuous Riesz bimorphism $\phi : L \times M \to N$ there is a continuous Riesz homomorphism $\phi_* : T \to N$ such that $\phi = \phi_* \circ b$ and $||\phi_*|| \leq ||\phi||$. Moreover, we require that ϕ_* is the unique Riesz homomorphism $\chi : T \to N$ with the property that $\phi = \chi \circ b$.



Remark 6.2. Later on we will see that $||\phi_*|| = ||\phi||$ and when the spaces are non-trivial, that $||b|| = 1$.

If the normed Riesz space (Banach lattice) tensor product exists, then it is unique as Riesz space and as normed space.

Theorem 6.3. *Let L and M be normed Riesz spaces (Banach lattices). Suppose (S, b) and (T, c) are normed Riesz space (Banach lattice) tensor products of L and M. Then there is a unique Riesz homomorphism $\phi : S \to T$ such that $\phi \circ b = c$. Moreover ϕ is invertible, $\phi^{-1} : T \to S$ is a Riesz homomorphism and ϕ is isometric. In particular, ϕ is an order isomorphism.*

Proof. The proof is similar to the proof of Theorem 4.2 on page 11. □

Theorem 6.4. *Let L and M be normed Riesz spaces (Banach lattices). If (T, b) is the normed Riesz space (Banach lattice) tensor product of L and M, then T is generated as Riesz space (Banach lattice) by elements $b(x, y)$.*

Proof. Let L and M be normed Riesz spaces (Banach lattices). Suppose (T, b) is the normed Riesz space (Banach lattice) tensor product of L and M. Let S be the Riesz subspace (Banach sublattice) of T generated by elements $b(x, y)$. Note that b maps into S. Let N be an arbitrary normed Riesz space (Banach lattice) and let $\phi : L \times M \to N$ be a continuous Riesz homomorphism. Let $\phi'_* : T \to N$ be the unique continuous Riesz homomorphism with $\phi = \phi'_* \circ b$. Note that $||\phi_*|| \leq ||\phi'_*|| \leq ||\phi||$. Let $\phi_* : S \to N$ be the restriction of ϕ'_* to S. Then ϕ_* is a Riesz homomorphism and $\phi = \phi_* \circ b$. Let $\psi : S \to N$ be any Riesz homomorphism such that $\phi = \psi \circ b$. Then $\phi_*(b(x, y)) = \psi(b(x, y))$, for all $x \in L$ and $y \in M$. Thus ϕ_* and ψ coincide on S. Hence $\phi_* = \psi$. We conclude that (S, b) is also the normed Riesz space (Banach lattice) tensor product of L and M. From Theorem 6.3 follows that $S = T$. This concludes our proof. □

Theorem 6.5. *Let L and M be Banach lattices. Suppose (T', b) is the normed Riesz space tensor product of L and M. Let T be the norm completion of T'. Then (T, b) is the Banach lattice tensor product of L and M.*

Proof. Let L and M be Banach lattices. Suppose (T', b) is the normed Riesz space tensor product of L and M. Let T be the norm completion of T'. Then by Theorem 3.27 on page 10 T is a Banach lattice. Note that $b : L \times M \to T$ is a continuous Riesz bimorphism. Let N be any Banach lattice and $\phi : L \times M \to N$ a continuous Riesz bimorphism. Then there exist a unique continuous Riesz homomorphism $\psi : T' \to N$ such that $\phi = \psi \circ b$. Let ϕ_* be the continuous extension of ψ to T. Then ϕ_* is a Riesz homomorphism and $\phi = \phi_* \circ b$. Let $\chi : T \to N$ be an arbitrary Riesz homomorphism such that $\phi = \chi \circ b$. Since b maps into T', we have that $\phi = \chi|_{T'} \circ b$. From the uniqueness of ψ follows that $\chi|_{T'} = \psi$. The continuous extensions of $\chi|_{T'}$ and ψ coincide thus $\chi = \psi_*$. Note that $||\phi_*|| = ||\psi|| \leq ||\phi||$ and $||b|| \leq 1$. We conclude that (T, b) is the Banach lattice tensor product of L and M. □

6.2 Construction of the normed Riesz space and Banach lattice tensor product via a free normed Riesz space.

Theorem 6.6. *Let L_1 and L_2 be normed Riesz spaces (Banach lattices). If $L_1 = 0$ or $L_2 = 0$, then $(0,0)$ is the normed Riesz space (Banach lattice) tensor product of L_1 and L_2. Suppose L_1 and L_2 are non-zero, let $S_i = \{x \in L_i : ||x|| = 1\}$, for $i \in \{1,2\}$. Consider the free normed Riesz space (FNRS($S_1 \times S_2$), ι) (free Banach lattice FBL($S_1 \times S_2$)) with Riesz norm $||\cdot||_F$. Let J be the intersection of all $||\cdot||_F$-norm closed ideals that contain*

$$\left\{\begin{array}{l} ||x+y||\,||z||\iota\left(\frac{x+y}{||x+y||},\frac{z}{||z||}\right)-||x||\,||z||\iota\left(\frac{x}{||x||},\frac{z}{||z||}\right)-||y||\,||z||\iota\left(\frac{y}{||y||},\frac{z}{||z||}\right),\\ x,y\in L_1\backslash\{0\}, z\in L_2\backslash\{0\},\ \text{and } x+y\neq 0,\\ ||\alpha x||\,||y||\iota\left(\frac{\alpha x}{||\alpha x||},\frac{y}{||y||}\right)-\alpha||x||\,||y||\iota\left(\frac{x}{||x||},\frac{y}{||y||}\right),\\ x\in L_1\backslash\{0\}, y\in L_2\backslash\{0\}, \alpha\in\mathbb{R}\backslash\{0\},\\ ||x||\,||y+z||\iota\left(\frac{x}{||x||},\frac{y+z}{||y+z||}\right)-||x||\,||y||\iota\left(\frac{x}{||x||},\frac{y}{||y||}\right)-||x||\,||z||\iota\left(\frac{x}{||x||},\frac{z}{||z||}\right),\\ x\in L_1\backslash\{0\}, y,z\in L_2\backslash\{0\},\ \text{and } y+z\neq 0,\\ ||x||\,||\alpha y||\iota\left(\frac{x}{||x||},\frac{\alpha y}{||\alpha y||}\right)-\alpha||x||\,||y||\iota\left(\frac{x}{||x||},\frac{y}{||y||}\right),\\ x\in L_1\backslash\{0\}, y\in L_2\backslash\{0\}, \alpha\in\mathbb{R}\backslash\{0\},\\ ||x||\,||y||\left|\iota\left(\frac{x}{||x||},\frac{y}{||y||}\right)\right|-||x||\,||y||\iota\left(\frac{|x|}{||x||},\frac{|y|}{||y||}\right),\\ x\in L_1\backslash\{0\}, y\in L_2\backslash\{0\}. \end{array}\right. \tag{5}$$

Let $T = FNRS(S_1 \times S_2)/J$, and let $q : FNRS(S_1 \times S_2) \to T$ be the quotient homomorphism, define $b : L_1 \times L_2 \to T$ through

$$b(x,y)=\left\{\begin{array}{ll} 0 & \text{if } x=0,\ \text{or } y=0,\\ q\left(||x||\,||y||\iota\left(\frac{x}{||x||},\frac{y}{||y||}\right)\right) & \text{otherwise.}\end{array}\right. \tag{6}$$

Then b is a continuous Riesz bimorphism and (T,b) is the normed Riesz space (Banach lattice) tensor product of L_1 and L_2. Moreover for any normed Riesz space (Banach lattice) M and any continuous Riesz bimorphism $\phi : L_1 \times L_2 \to M$, the induced continuous Riesz homomorphism $\phi_* : T \to M$ that satisfies $\phi = \phi_* \circ b$ also satisfies $||\phi_*|| = ||\phi||$. Furthermore if L_1 and L_2 are non-trivial then $||b|| = 1$.

Proof. Since the proof of the Banach lattice case is similar to the proof of the normed Riesz space case, we only do the latter. Let L_1 and L_2 be normed Riesz spaces. The case that L_1 or L_2 is 0 is trivial, so suppose that both L_1 and L_2 are non-trivial. Let S_1, S_2, J, T, q and b be as in the theorem. We will show that b is a Riesz bimorphism and that (T,b) is the normed Riesz space tensor product of L_1 and L_2.

Claim 6.7. *b is a continuous Riesz bimorphism.*

Proof of Claim 6.7. Note that $b(x,y) = 0$, if $x = 0$ or $y = 0$. Let $\alpha \in \mathbb{R}$. If $\alpha = 0$, then $b(\alpha x, y) = b(0,y) = 0 = \alpha b(x,y)$. Suppose $\alpha \neq 0$. Clearly, if $x = 0$ or $y = 0$ or both, then $b(\alpha x, y) = \alpha b(x,y)$. If $x \neq 0, y \neq 0$ and $\alpha \neq 0$, then

$$\begin{aligned} b(\alpha x,y) &= q\left(||\alpha x||\,||y||\iota\left(\frac{\alpha x}{||\alpha x||},\frac{y}{||y||}\right)\right)\\ &= q\left(\alpha||x||\,||y||\iota\left(\frac{x}{||x||},\frac{y}{||y||}\right)\right)\\ &= \alpha q\left(||x||\,||y||\iota\left(\frac{x}{||x||},\frac{y}{||y||}\right)\right)\\ &= \alpha b(x,y). \end{aligned}$$

Let $x, y \in L_1, z \in L_2$. If $z = 0$, then $b(x+y,z) = 0 = b(x,z) + b(y,z)$. If $z \neq 0$, and $x = 0$ or $y = 0$ or both are 0, then, clearly, $b(x+y,z) = b(x,z) + b(y,z)$. If $z \neq 0, x \neq 0$ and $y \neq 0$ but $x + y = 0$, then $x = -y$. Thus $b(x,z) = b(-y,z) = -b(y,z)$. Hence $b(x+y,z) = 0 = b(x,z) + b(y,z)$. Finally, if $x \neq 0, y \neq 0, z \neq 0$ and $x + y \neq 0$, then

$$\begin{aligned} b(x+y,z) &= q\left(||x+y||\,||z||\iota\left(\tfrac{x+y}{||x+y||}, \tfrac{z}{||z||}\right)\right) \\ &= q\left(||x||\,||z||\iota\left(\tfrac{x}{||x||}, \tfrac{z}{||z||}\right) + ||y||\,||z||\iota\left(\tfrac{y}{||y||}, \tfrac{z}{||z||}\right)\right) \\ &= q\left(||x||\,||z||\iota\left(\tfrac{x}{||x||}, \tfrac{z}{||z||}\right)\right) + q\left(||y||\,||z||\iota\left(\tfrac{y}{||y||}, \tfrac{z}{||z||}\right)\right) \\ &= b(x,z) + b(y,z). \end{aligned}$$

Hence $b(\cdot, z)$ is linear, for all $z \in L_2$. Likewise $b(x, \cdot)$ is linear, for all $x \in L_1$. Thus b is bilinear. Let $x \in L_1, y \in L_2$. Suppose $x = 0$ or $y = 0$ or both, then $|b(x,y)| = 0 = b(|x|,|y|)$. Suppose $x \neq 0$ and $y \neq 0$. Then

$$\begin{aligned} |b(x,y)| &= \left|q\left(||x||\,||y||\iota\left(\tfrac{x}{||x||}, \tfrac{y}{||y||}\right)\right)\right| \\ &= q\left(||x||\,||y||\left|\iota\left(\tfrac{x}{||x||}, \tfrac{y}{||y||}\right)\right|\right) \\ &= q\left(||x||\,||y||\iota\left(\tfrac{|x|}{||x||}, \tfrac{|y|}{||y||}\right)\right) \\ &= b(|x|,|y|). \end{aligned}$$

Thus b is a Riesz bimorphism. Clearly b is continuous. □

Let N be an arbitrary normed Riesz space and let $\phi : L_1 \times L_2 \to N$ be a continuous Riesz bimorphism. Let $\phi_1 : S_1 \times S_2 \to N$ be the restriction of ϕ to $S_1 \times S_2$. Note that ϕ_1 is a bounded map. Let $\psi : \mathrm{FBL}(S_1 \times S_2) \to N$ be the unique Riesz homomorphism that satisfies $\phi_1 = \psi \circ \iota$ with the property $||\psi|| = ||\phi_1|| = ||\phi||$. We will prove that $J \subset \ker(\psi)$. Since $\ker(\psi)$ is a $||\cdot||_F$-norm closed ideal, we only have to prove that the set that generates J is contained in $\ker(\psi)$.
Let $\alpha \in \mathbb{R}\backslash\{0\}, x \in L_1\backslash\{0\}, y \in L_2\backslash\{0\}$. Then

$$\begin{aligned} &\psi\left(||\alpha x||\,||y||\iota\left(\tfrac{\alpha x}{||\alpha x||}, \tfrac{y}{||y||}\right) - \alpha||x||\,||y||\iota\left(\tfrac{x}{||x||}, \tfrac{y}{||y||}\right)\right) \\ =\;& ||\alpha x||\,||y||\psi\left(\iota\left(\tfrac{\alpha x}{||\alpha x||}, \tfrac{y}{||y||}\right)\right) - \alpha||x||\,||y||\psi\left(\iota\left(\tfrac{x}{||x||}, \tfrac{y}{||y||}\right)\right) \\ =\;& ||\alpha x||\,||y||\phi_1\left(\tfrac{\alpha x}{||\alpha x||}, \tfrac{y}{||y||}\right) - \alpha||x||\,||y||\phi_1\left(\tfrac{x}{||x||}, \tfrac{y}{||y||}\right) \\ =\;& \phi(\alpha x, y) - \alpha\phi(x,y) \\ =\;& 0. \end{aligned}$$

Thus $||\alpha x||\,||y||\iota\left(\frac{\alpha x}{||\alpha x||}, \frac{y}{||y||}\right) - \alpha||x||\,||y||\iota\left(\frac{x}{||x||}, \frac{y}{||y||}\right) \in \ker(\psi)$, for all $x \in L_1\backslash\{0\}, y \in L_2\backslash\{0\}$ and $\alpha \in \mathbb{R}\backslash\{0\}$. Likewise $||x||\,||\alpha y||\iota\left(\frac{x}{||x||}, \frac{\alpha y}{||\alpha y||}\right) - \alpha||x||\,||y||\iota\left(\frac{x}{||x||}, \frac{y}{||y||}\right) \in \ker(\psi)$, for all $x \in L_1\backslash\{0\}, y \in L_2\backslash\{0\}$ and $\alpha \in \mathbb{R}\backslash\{0\}$.
For all $x, y \in L_1\backslash\{0\}$ and $z \in L_2\backslash\{0\}$ with $x + y \neq 0$ we have

$$\begin{aligned} &\psi\left(||x+y||\,||z||\iota\left(\tfrac{x+y}{||x+y||}, \tfrac{z}{||z||}\right) - ||x||\,||z||\iota\left(\tfrac{x}{||x||}, \tfrac{z}{||z||}\right) - ||y||\,||z||\iota\left(\tfrac{y}{||y||}, \tfrac{z}{||z||}\right)\right) \\ =\;& ||x+y||\,||z||\psi\left(\iota\left(\tfrac{x+y}{||x+y||}, \tfrac{z}{||z||}\right)\right) - ||x||\,||z||\psi\left(\iota\left(\tfrac{x}{||x||}, \tfrac{z}{||z||}\right)\right) - ||y||\,||z||\psi\left(\iota\left(\tfrac{y}{||y||}, \tfrac{z}{||z||}\right)\right) \\ =\;& ||x+y||\,||z||\phi_1\left(\tfrac{x+y}{||x+y||}, \tfrac{z}{||z||}\right) - ||x||\,||z||\phi_1\left(\tfrac{x}{||x||}, \tfrac{z}{||z||}\right) - ||y||\,||z||\phi_1\left(\tfrac{y}{||y||}, \tfrac{z}{||z||}\right) \\ =\;& \phi(x+y,z) - \phi(x,z) - \phi(y,z) \\ =\;& 0. \end{aligned}$$

Thus $||x+y||\,||z||\iota\left(\frac{x+y}{||x+y||}, \frac{z}{||z||}\right) - ||x||\,||z||\iota\left(\frac{x}{||x||}, \frac{z}{||z||}\right) - ||y||\,||z||\iota\left(\frac{y}{||y||}, \frac{z}{||z||}\right) \in \ker(\psi)$. Likewise, for all $x \in L_1\backslash\{0\}$ and $y, z \in L_2\backslash\{0\}$ with $y + z \neq 0$, we have that $||x||\,||y+z||\iota\left(\frac{x}{||x||}, \frac{y+z}{||y+z||}\right) -$

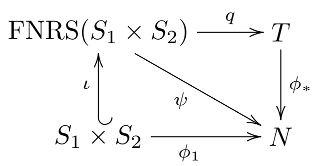
$||x||\,||y||\iota\left(\frac{x}{||x||},\frac{y}{||y||}\right) - ||x||\,||z||\iota\left(\frac{x}{||x||},\frac{z}{||z||}\right) \in \ker(\psi)$.
For all $x \in L_1\backslash\{0\}$ and $y \in L_2\backslash\{0\}$ we have

$$\begin{aligned} & \psi\left(||x||\,||y||\left|\iota\left(\tfrac{x}{||x||},\tfrac{y}{||y||}\right)\right| - ||x||\,||y||\iota\left(\tfrac{|x|}{||x||},\tfrac{|y|}{||y||}\right)\right) \\ = \; & ||x||\,||y||\left|\psi\left(\iota\left(\tfrac{x}{||x||},\tfrac{y}{||y||}\right)\right)\right| - ||x||\,||y||\psi\left(\iota\left(\tfrac{|x|}{||x||},\tfrac{|y|}{||y||}\right)\right) \\ = \; & ||x||\,||y||\left|\phi_1\left(\tfrac{x}{||x||},\tfrac{y}{||y||}\right)\right| - ||x||\,||y||\phi_1\left(\tfrac{|x|}{||x||},\tfrac{|y|}{||y||}\right) \\ = \; & |\phi(x,y)| - \phi(|x|,|y|) \\ = \; & 0. \end{aligned}$$

Thus $||x||\,||y||\left|\iota\left(\frac{x}{||x||},\frac{y}{||y||}\right)\right| - ||x||\,||y||\iota\left(\frac{|x|}{||x||},\frac{|y|}{||y||}\right) \in \ker(\psi)$. It follows that $J \subset \ker(\psi)$.
Define $\phi_* : T \to N$ through $\phi_*(q(x)) = \psi(x)$, for $x \in \mathrm{FBL}(S_1 \times S_2)$. Then ϕ_* is a well defined Riesz homomorphism. We have

$$\begin{aligned} \phi_*(b(x,y)) &= \phi_*\left(q\left(||x||\,||y||\iota\left(\tfrac{x}{||x||},\tfrac{y}{||y||}\right)\right)\right) \\ &= ||x||\,||y||\psi\left(\iota\left(\tfrac{x}{||x||},\tfrac{y}{||y||}\right)\right) \\ &= ||x||\,||y||\phi_1\left(\tfrac{x}{||x||},\tfrac{y}{||y||}\right) \\ &= \phi(x,y), \end{aligned}$$

for all $x \in L_1\backslash\{0\}$ and $y \in L_2\backslash\{0\}$. If $x \in L_1, y \in L_2$ and $x = 0$ or $y = 0$ or both, then $\phi_*(b(x,y)) = \phi_*(q(0)) = \psi(0) = 0 = \phi(x,y)$. So $\phi = \phi_* \circ b$.
Suppose $\chi : T \to N$ is a Riesz homomorphism that satisfies $\phi = \chi \circ b$. In particular $\phi_1 = \chi \circ q \circ \iota$. From the uniqueness statement in the definition of a free normed Riesz space follows that $\chi \circ q : \mathrm{FNRS}(S_1 \times S_2) \to N$ is equal to ψ. Thus $\chi(q(x)) = \psi(x) = \phi_*(q(x))$, for all $x \in \mathrm{FBL}(S_1 \times S_2)$. Hence $\chi = \phi_*$.
Note that $||b(x,y)|| = ||q(\iota(x,y))|| \leq ||\iota(x,y)|| = 1 = ||x||\,||y||$, for all $x \in S_1, y \in S_2$. Thus $||b|| \leq 1$. For all $x \in \mathrm{FBL}(S_1 \times S_2)$ of norm one, we have $||\phi_*(q(x))|| = ||\psi(x)||$ thus $||\phi_*|| = ||\psi|| = ||\phi_1|| = ||\phi||$. This concludes our proof that (T,b) is the normed Riesz space tensor product of L_1 and L_2.
Note that b is non-trivial. Thus $||b|| > 0$. We have already seen that $||b|| \leq 1$. Suppose $||b|| = 1-\varepsilon < 1$ for some $0 < \varepsilon < 1$. Then for all $x \in S_1, y \in S_2$ we have that $||b(x,y)|| = ||b_*(b(x,y))|| \leq ||b_*||\,||b(x,y)|| \leq (1-\varepsilon)||b_*|| = (1-\varepsilon)||b||$, and that is a contradiction. We conclude that $||b|| = 1$. This concludes the proof of the theorem.



□

Remark 6.8. D.H. Fremlin proved also the following fact for Banach lattices [4, Theorem 1E(iii)]: let L, M and N be Banach lattices and let (T,b) be the Banach lattice tensor product of L and M. Then there is a one-to-one norm preserving correspondence between the continuous positive bimorphisms $\phi : L \times M \to N$ and the continuous positive linear maps $\phi_* : T \to N$ such that $\phi = \phi_* \circ b$. Moreover ϕ is a continuous Riesz bimorphism if and only if ϕ_* is a continuous Riesz homomorphism. We could not find a proof for this fact with the new construction.

Theorem 6.9. *Let L', L, M' and M be normed Riesz spaces (Banach lattices) with $L' \subset L$ and $M' \subset M$. Let (T',b') be the normed Riesz space (Banach lattice) tensor product of L' and M' and let (T,b) be the normed Riesz space (Banach lattice) tensor product of L and M. Then there is an injective continuous Riesz homomorphism $\iota : T' \to T$ such that $\iota(b'(x,y)) = b(x,y)$, for all $x \in L'$ and $y \in M'$. In particular ι is bipositive.*

Proof. The proof is similar to the proof of Theorem 4.8 on page 13. Note that $||\iota|| \leq ||b|| \leq 1$. Hence ι is continuous. □

Remark 6.10.
1. For Banach lattices L and M we denote the Banach lattice tensor product T by $L\widehat{\otimes}M$ and the bimorphism b by $\otimes$.
2. From the construction follows clearly that $L\widehat{\otimes}M$ is isomorphic as Banach lattice to $M\widehat{\otimes}L$.

References

[1] C.D. Aliprantis & O. Burkinshaw, *Positive Operators*, Academic Press, Orlando, 1985

[2] C.D. Aliprantis & R. Tourky, *Cones and Duality*, American Mathematical Society, Providence, Rhode Island, 2007

[3] D.H. Fremlin, *Tensor Products of Archimedean Vector Lattices*, American J. Math. 94, 777-798, 1972

[4] D.H. Fremlin, *Tensor Products of Banach Lattices*, Math. Ann. 211, 87-106 (1974), 1974, http://link.springer.com/content/pdf/10.1007%2FBF01344164.pdf

[5] D.H. Fremlin, *Measure Theory*, Volume 3, Measure Algebras, Reader in Mathematics, University of Essex, 2002

[6] O. van Gaans, *Seminorms on Ordered Vector Spaces*, Phd. Thesis Radboud University of Nijmegen, Nijmegen, 1999

[7] O. van Gaans & A. Kalauch, *Tensor Products of Archimedean Partially Ordered Vector Spaces*, Report MI-2010-01, Mathematical Institute, Leiden University, Leiden, 2010, http://link.springer.com/content/pdf/10.1007%2Fs11117-010-0085-5.pdf

[8] O. van Gaans & A. Kalauch, *Directed ideals in partially ordered vector spaces*, Indagationes mathematicae (to appear), 2013

[9] O. van Gaans, A. Kalauch & B. Lemmens, *Riesz completions, functional representations, and anti-lattices*, Pre-publication, 2012, http://www.kent.ac.uk/smsas/personal/bl81/klvgfinal.pdf

[10] M.B.J.G. van Haandel, *Completions in Riesz Space Theory*, Ph.D. thesis, Radboud University of Nijmegen, Nijmegen, 1993

[11] Peter Meyer-Nieberg, *Banach Lattices*, Springer-Verlag, Berlin - New York, 1991

[12] W.A.J. Luxemburg & A.C. Zaanen, *Riesz spaces, Volume I*, North-Holland Publishing Company, Amsterdam - London, 1971

[13] B. de Pagter, Prepublication (no title), Delft University of Technology, Delft, 2012

[14] B. de Pagter and A.W. Wickstead, *Free and Projective Banach Lattices*, http://arxiv.org/pdf/1204.4282v1.pdf

[15] J. van Waaij, *Suprema in Spaces of Operators (Dutch)*, Bachelor thesis, Leiden University, Leiden, 2011, http://www.math.leidenuniv.nl/scripties/BachVanWaaij.pdf

Index