On the decomposition of quasi-martingales on Riesz spaces

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Outline

1. Stochastic Processes
    - Quasi-Martingales
    - Riesz Spaces

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Stochastic Processes

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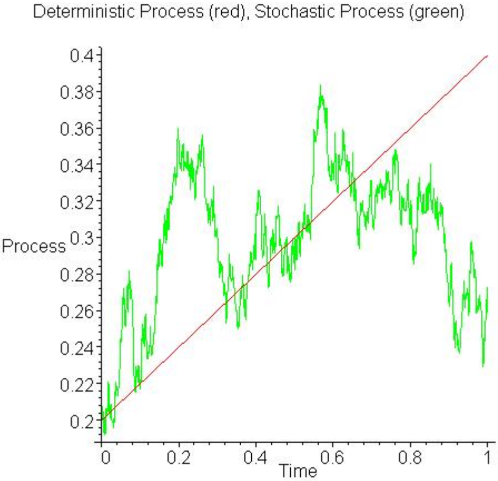

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1. Stochastic Processes
   - Quasi-Martingales
   - Riesz Spaces
2. Stochastic Processes on Riesz Spaces
   - Preliminaries
   - Continuous time Stochastic processes and right continuity
   - Quasi-Martingales on Riesz Spaces

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- In the classical setting: Definition:
  A process $X = X_t$ is said to be a quasi-martingale if and only if there exists a constant M such that

$$\sup_{\{t_1<t_2<\cdots<t_n\}\in\mathbb{R}^+} \sum_{i=1}^{n} \mathbb{E}\left[|X_{t_i} - \mathbb{E}[X_{t_{i+1}}|\mathcal{F}_{t_i}]|\right] \leq M.$$

  A quasi-martingale $X = X_t$ is said to be a quasi-potential if

$$\lim_{t\to\infty} \mathbb{E}[|X_t|] = 0.$$

Examples

- Sub- and super-martingales.

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- State of set of ISI journals, as constructed by Egghe.

Outline

Riesz Spaces

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- For our purposes, elements of our Riesz space are functions (or operators).

Outline

Dictionary

Classical Setting	Riesz Space Setting
$(\Omega.\mathcal{F}, P)$	E, Dedekind complete
$\mathbb{E}[\cdot \,\vert\Sigma]$	T_i
$\Sigma_1 \subset \Sigma_2 \subset \ldots$	$(T_i)_{i\in\mathbb{R}}$ s.t $T_j T_i = T_i = T_i T_j$ where $i \leq j$

Background

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- Conditional expectations on Riesz spaces:
  Definition:
  Let E be a Riesz space with weak order unit. A positive order continuous projection T on E with range, $R(T)$, a Dedekind complete Riesz subspace of E, is called a conditional expectation if Te is a weak order unit of E for each weak order unit e in E.

An Example

Consider $\mathcal{L}^1(\Omega, \mathcal{F}, P)$ with sub-σ-algebra Σ. Then $\mathcal{L}^1(\Omega, \mathcal{F}, P)$ is a Riesz space. In this case, $T = \mathbb{E}[\cdot\,|\Sigma]$ and the weak order unit is $\mathbf{1}$.

Definition

Let E be a Riesz space with weak order unit. A filtration on E is a family of conditional expectations, $(T_i)_{i\in\mathbb{N}}$, on E with

$$T_i T_j = T_i = T_j T_i, \quad \text{for all} \quad i \leq j.$$

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(ii) We say that the filtration, $(T_t)_{t\in[0,\infty)}$, is **right continuous** if $R(T_s) = \bigcap_{t>s} R(T_t)$ for all $s \in [0,\infty)$ (Grobler).

A note on right continuity

If E is T-universally complete, there is a conditional expectation operator T_{s^+} with range $\bigcap_{t>s} R(T_t)$ for each $s \in [0, \infty)$.

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If E is T-universally complete, there is a conditional expectation operator T_{s^+} with range $\bigcap_{t>s} R(T_t)$ for each $s \in [0, \infty)$.

In this setting, $(T_t)_{t\in[0,\infty)}$ is right continuous if and only if

$$T_{s^+} = T_s \text{ for all } s \in [0, \infty).$$

Definition

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(a) (f_t) is **right continuous** if $f_t \to f_s$ as $t \downarrow s$.

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(b) (f_t) is **sequentially right continuous** if for each $s \in [0,\infty)$ and each sequence $(t_i)_{i\in\mathbb{N}}$ with $t_i \downarrow_i s$ we have that $f_{t_i} \to f_s$ as $i \to \infty$.

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(c) (f_t) is T**-sequentially right continuous** if for each $s \in [0,\infty)$ and each sequence $(t_i)_{i\in\mathbb{N}}$ with $t_i \downarrow_i s$ we have that $T|f_{t_i} - f_s| \to 0$ as $i \to \infty$.

On continuity

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right continuity$\Rightarrow$ sequential right continuity$\Rightarrow$ T-sequential right continuity.
- Under the usual conditions, every martingale is sequentially right continuous.

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The "Usual Conditions"

'Let E be a T-universally complete Riesz space with weak order unit where T is a strictly positive conditional expectation operator. Let $(T_t)_{t\in[0,\infty)}$ be a filtration on E, with $T_tT = T = TT_t$, $t \in [0,\infty)$.'

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(i) (f_t) is adapted to (T_t);

(ii) there exists a constant M such that

$$\sup_{(t_1,t_2,\ldots,t_{n+1})\in\Pi}\sum_{i=1}^{n} T|f_{t_i} - T_{t_i} f_{t_{i+1}}| \leq M.$$

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Π is the collection of all finite sequences of real numbers, $(t_1, t_2, \ldots, t_{n+1}), n \in \mathbb{N}$, with $0 \leq t_1 < t_2 < t_3 < \cdots < t_{n+1}$.

Definition

If $(f_t)_{t\in[0,\infty)}$ is a T-quasi-martingale, then we say $(f_t)_{t\in[0,\infty)}$ is a T-quasi-potential if

$$\lim_{t\to\infty} T|f_t| = 0.$$

Decompositions of Quasi-Martingales

Theorem (Riesz Decomposition Theorem)

Under the usual conditions, every T-quasi-martingale can be written uniquely as the sum of a martingale and a T-quasi-potential.

Decompositions of Quasi-Martingales

Theorem

Under the usual conditions, if $(X_t)_{t\in[0,\infty)} \subset E$ is a T-sequentially right continuous T-quasi-potential adapted to $(T_t)_{t\in[0,\infty)}$ then there exist positive potentials X_t^p, X_t^m such that

$$X_t = X_t^p - X_t^m \qquad \text{for all } t \in [0,\infty). \tag{1}$$

Decompositions of Quasi-Martingales

Theorem

Under the usual conditions, if $(X_t)_{t\in[0,\infty)} \subset E$ is a T-sequentially right continuous T-quasi-martingale then $(X_t)_{t\in[0,\infty)}$ can be decomposed into the sum of a martingale with the difference of two positive potentials.

An application of the Riesz space theory

Consider $(\Omega, \mathcal{F}, \mu)$, $\mathcal{F}_0 \subset \mathcal{F}$ and $t, t_n \in [0, \infty)$.
E: RS of all $\mathcal{F}$-msb f on Ω s.t $\mathbb{E}[|f||\mathcal{F}_0]$ exists.
$\mathcal{F}_t$ a right continuous with $\mathcal{F}_0 \subset \mathcal{F}_t \subset \mathcal{F}$
$X_t \in E$ a 'conditional expectation' quasi-martingale i.e

$$\sum_{i=1}^{n} \mathbb{E}[|X_{t_i} - \mathbb{E}[X_{t_{i+1}}|\mathcal{F}_{t_i}]|\mathcal{F}_0] \leq M.$$

If, for $t_n \downarrow t$, we have

$$\lim_{n\to\infty} \mathbb{E}[|X_{t_n} - X_t||\mathcal{F}_0] = 0,$$

then there exists a martingale Y_t and positive super-martingales, $X_t^{\pm}$ so that $X_t = Y_t + X_t^{+} - X_t^{-}$ and $\lim_{t\to\infty} \mathbb{E}[X_t^{\pm}|\mathcal{F}_0] = 0$.

Thank you

Any questions?