

A NON-GIVEN TALK ON TURING INSTABILITY OF A REACTION-DIFFUSION SYSTEM WITH OBSTACLES

MARTIN VÄTH

Due to the evacuation of the building, the talk corresponding to these notes was never given. These are *not* the transparencies of the talk (which would not make sense without any explanation) but instead a summary of a modified version of the talk:

In order to motivate you to read these notes, I keep them short and have omitted the new instability result in preparation [5] (although this would reveal many connections with Wolfgang Arendt’s talk, in particular, it would apply variations of some of the regularity results he has mentioned) and just concentrate on the bifurcation problem.

1. FORCED POSITIVITY

The bifurcation problem studied in the next section is actually a problem with “forced positivity”. For those readers working mainly with ordered Banach spaces, this may appear a bit unusual, although there are strong relations.

In several talks on this conference—e.g. in those of Bas Lemmens (joint work with Roger Nussbaum) or Horst R. Thieme—we learnt about properties of positively homogeneous order-preserving cone-maps and that it is surprising how much the behaviour of such maps differs from that of the linear case.

In the case of “forced positivity” (which are actual variational inequalities as we will see) this is even more surprising, since this case is even “ruled” by a linear operator. (However, our maps are usually not order-preserving.)

To get to the right track in the functional analytic setting, we consider first a classical bifurcation problem in some real Hilbert space \mathbb{H} , say

$$(1.1) \quad u = F(\lambda, u)$$

where $F: (0, \infty) \times \mathbb{H} \rightarrow \mathbb{H}$ is completely continuous with $F(\cdot, 0) = 0$. We assume that F has the form

$$F(u) = \lambda^{-1}(Au + G(\lambda, u))$$

where A is linear and $\|G(\lambda, u)\| = o(\|u\|)$ uniformly for λ close to some considered point $\lambda_0 > 0$. (We will later consider a more general dependence $A(\lambda)$ instead of the particular form $\lambda^{-1}A$, but the latter makes things clearer in the moment.) Then (1.1) becomes

$$\lambda u = Au + G(\lambda, u)$$

The linear operator A is compact, since it is the derivative of a completely continuous operator. For simplicity, let us assume in this section that A is symmetric.

Recall that $\lambda_0 > 0$ is a (local) bifurcation point of (1.1) if each neighborhood of $(\lambda_0, 0)$ in $\mathbb{R} \times \mathbb{H}$ contains nontrivial solutions (λ, u) of (1.1) (nontrivial means $u \neq 0$). It is well-known that a necessary condition for $\lambda_0 > 0$ to be a bifurcation point is that

The conference was visited in the framework of a DFG project (Az. VA 206/2-1). Financial support by the DFG is gratefully acknowledged.

it is a spectral value (that is, an eigenvalue) of A . The famous Rabinowitz bifurcation theorem [12] states that this necessary condition is also sufficient if the multiplicity of this eigenvalue is odd and, moreover, the bifurcation is then “global”, that is, there is a connected set $\mathcal{C} \subseteq (0, \infty) \times \mathbb{H}$ of nontrivial solutions which meets $(\lambda_0, 0)$ and which is either unbounded or which meets either $\{0\} \times \mathbb{H}$ or $(\lambda_1, 0)$ for another bifurcation point λ_1 . Recall that the condition about odd eigenvalues cannot be dropped: There are counterexamples with multiplicity 2 and without any (local) bifurcation.

In the case of “forced positivity” a “corresponding” result has 2 big surprises. By “positivity” we mean here that we are given a closed convex cone $K \subseteq \mathbb{H}$ (in the applications we have in mind, K is *not* pointed, i.e., it does not induce a partial order). By “forced” we mean here that we use the metric projection $P_K: \mathbb{H} \rightarrow K$, that is, $P_K(u)$ is that element of K with closest distance to u , to force F to be a self-map of K . In other words, we consider the problem

$$(1.2) \quad u = P_K(F(\lambda, u))$$

which by our hypothesis about F means

$$u = P_K(\lambda^{-1}(Au + G(\lambda, u)))$$

or, equivalently (recall that P_K is positively homogeneous),

$$\lambda u = P_K(Au + G(\lambda, u)).$$

Using the well-known characterization of the metric projection P_K in a real Hilbert space, we note that (1.2) is equivalent to the variational inequality

$$u \in K, \quad \langle u - F(\lambda, u), \varphi - u \rangle \geq 0 \text{ for all } \varphi \in K.$$

To get an idea about the problem (1.2), we consider the “linearization” $G = 0$

$$u = P_K(\lambda^{-1}Au),$$

that is

$$(1.3) \quad \lambda u = P_K(Au).$$

In this case, the right-hand side $P_K A$ is a positively homogeneous map into the cone K , and for any reasonable generalization of the term “derivative”, $P_K A$ is its own derivative at 0 (although it is a nonlinear operator, of course, if K is not a linear subspace). In this sense (1.3) is really the “linearization” of (1.2).

Clearly, if $\lambda = \lambda_0$ is an eigenvalue of A with at least one corresponding eigenvector in K then it is also an “eigenvalue” of the problem (1.3), that is, (1.3) has a nontrivial solution (namely the eigenvector of A in K).

The first surprise about “forced positivity” is that the Rabinowitz type result holds without any hypotheses about multiplicity of such an eigenvalue: If λ_0 is an eigenvalue of A with a least one eigenvector in the *interior* of K and at least one eigenvector not in $(-K)$ then λ_0 is automatically a bifurcation point of (1.2), even a global bifurcation point in the sense mentioned earlier: The first results in this direction were obtained by a so-called homotopy method in [6, 7], and by degree methods in [9]. In many applications, K has no interior, so it is crucial that by a refinement of the techniques the “interior” can be replaced by a certain “pseudo-interior” [11]; the latter can be relaxed even more,

and parts of this result hold also without symmetry or compactness of A and even in a Banach space, as shown recently by the author [15].

The second surprise about “forced positivity” is that *typically* there are eigenvalues of (1.3) which are *not* eigenvalues of A . In fact, between *any* pair $\lambda_1 < \lambda_2$ of interior eigenvalues of A (by an interior eigenvalue we mean that the hypothesis mentioned in the previous paragraph is satisfied) there is an eigenvalue λ_0 of (1.3) (with $\lambda_1 < \lambda_0 < \lambda_2$), even more, λ_0 is a global bifurcation point of (1.3) in the sense mentioned earlier. Note that *typically* λ_0 is *not* an eigenvalue of A : Roughly speaking, (1.3) has at least twice as many positive eigenvalues as A . (In fact, there are even finite-dimensional examples where (1.3) has a whole interval of eigenvalues; it seems to be unknown whether such examples occur in real-world applications.)

All results mentioned above follow from the papers cited above, e.g. from [15]. As mentioned above, the latter paper also deals partially with nonsymmetric A and with operators in Banach spaces (instead of Hilbert spaces): It may be surprising to the reader that even in Banach spaces the problem (1.2) becomes a variational inequality. In fact, one “just” has to replace the scalar product by an appropriate semi-inner product, see [15] for details. (Recall that we heard talks about semi-inner products on this conference, e.g. by Eder Kikianty.)

The problem we discuss now is actually one of the simplest special cases with non-symmetric A , namely the case that $\mathbb{H} = \mathbb{H}_0 \times \mathbb{H}_0$ and A is a 2×2 “block matrix” with entries all being scalar multiples of the same symmetric compact operator A_0 . Instead of a dependence from $\lambda > 0$, we will consider a certain dependence from $(d_1, d_2) \in \mathbb{R}_+^2$.

2. TURING INSTABILITY AND OBSTACLES

Consider the reaction-diffusion system

$$(2.1) \quad \begin{aligned} \frac{du}{dt} &= d_1 \Delta u + f_1(u, v), \\ \frac{dv}{dt} &= d_2 \Delta v + f_2(u, v) \end{aligned}$$

with C^1 functions f_i in a bounded Lipschitz domain $\Omega \subseteq \mathbb{R}^N$ with Neumann boundary conditions

$$(2.2) \quad \frac{\partial u}{\partial n} = \frac{\partial v}{\partial n} = 0 \quad \text{on } \partial\Omega.$$

In a biochemical interpretation, u and v denote the difference of the concentration of two substances to some (constant in time and space) equilibrium $(0, 0)$. In particular, this means $f_i(0, 0) = 0$, and negative values of u and v have a sane biochemical interpretation. Linearizing (f_1, f_2) at $(0, 0)$, we rewrite (2.1) in the form

$$\begin{aligned} \frac{du}{dt} &= d_1 \Delta u + b_{11}u + b_{12}v + g_1(u, v), \\ \frac{dv}{dt} &= d_2 \Delta v + b_{21}u + b_{22}v + g_2(u, v) \end{aligned}$$

where $g_i(0, 0) = 0$, $g'_i(0, 0) = 0$. We assume that $b_{11} > 0$ and that the matrix $B = (b_{ij}) = (f_1, f_2)'(0, 0)$ has its spectrum in the left half-plane, that is, without the diffusion term

($d_1 = d_2 = 0$) the problem (2.1) is linearly stable; this assumption is equivalent to

$$b_{11} > 0 > b_{22}, \quad b_{12}b_{21} < 0, \quad b_{11} + b_{22} < 0 < b_{11}b_{22} - b_{12}b_{21}.$$

In order to work with completely continuous operators in $W^{1,2}(\Omega)$, we assume in case $N \geq 2$ that f_i and thus g_i are subject to the so-called subcritical growth condition

$$|f_i(u, v)| \leq c \cdot (1 + |u| + |v|)^p$$

for some $p > 0$ (in case $N \geq 3$, we require $p < \frac{N}{N-2}$).

It was the famous Alan Turing [13] who observed that, although $(0, 0)$ is linearly stable for the reaction part ($d_1 = d_2 = 0$) of (2.1) as well as for the diffusion part ($f_1 = f_2 = 0$), it can fail to be stable for the combination due to the nonsymmetry of the problem; this instability can lead to stationary (spatially non-constant) patterns which can be observed in real live (e.g. in the colouring of zebras).

Making this quantitatively more precise is an exercise in applying spectral theory of symmetric operators in Hilbert spaces: Let us assume that we know the eigenvalues of $(-\Delta)$ on Ω with Neumann boundary conditions

$$(2.3) \quad \frac{\partial u}{\partial n} = 0 \quad \text{on } \partial\Omega.$$

Recall that these eigenvalues form a sequence $0 = \kappa_0 < \kappa_1 < \dots \rightarrow \infty$. We are first interested in nontrivial stationary solutions of the linearization of (2.1), i.e. in

$$(2.4) \quad \begin{aligned} 0 &= d_1 \Delta u + b_{11}u + b_{12}v, \\ 0 &= d_2 \Delta v + b_{21}u + b_{22}v \end{aligned}$$

with boundary conditions (2.2). Some spectral calculus shows that for $d_1, d_2 > 0$ this problem has a nontrivial solution $(u, v) \neq (0, 0)$ if and only if (d_1, d_2) belongs to at least one of the hyperbolas

$$(2.5) \quad \begin{aligned} C_n &= \{(d_1, d_2) \in \mathbb{R}_+^2 : (\kappa_n d_1 - b_{11})(\kappa_n d_2 - b_{22}) = b_{12}b_{21}\} \\ &= \left\{ (d_1, d_2) \in \mathbb{R}_+^2 : d_2 = \frac{b_{12}b_{21}/\kappa_n^2}{d_1 - b_{11}/\kappa_n} + \frac{b_{22}}{\kappa_n} \right\}. \end{aligned}$$

This sequence of hyperbolas automatically looks qualitatively as in Figure 1: The hyperbolas have vertical asymptotes (at $d_1 = \frac{b_{11}}{\kappa_n}$) which accumulate to 0 and a joint common tangent through $(0, 0)$ whose slope is larger than 1 (typically much larger).

Moreover, it can be calculated that $(u, v) = (0, 0)$ in the system (2.1), (2.3) is linearly stable if and only if (d_1, d_2) lies to the right/under every C_n , that is, if (d_1, d_2) belongs to the right part D_S of the “envelope” of the sets C_n : For obvious reasons, we call D_S the “domain of stability”.

From a biological point of view, it is very bad that D_S is so large, because it means that the effect described by Turing occurs only rarely if one starts close to the equilibrium: A necessary condition is that the diffusion speed d_2 of v must be much larger than the diffusion speed d_1 of u .

It is the aim of these notes to observe that the situation is completely different in case of “forced positivity” (so-called “obstacles”): We think of the case that in some place of $\bar{\Omega}$ there is a source which becomes only active if the concentration of v would go under 0, thus ensuring that $v \geq 0$ on this place. (The case of a sink and $v \leq 0$ would lead to analogous results.) The surprising result is that such a source/sink, no matter how small

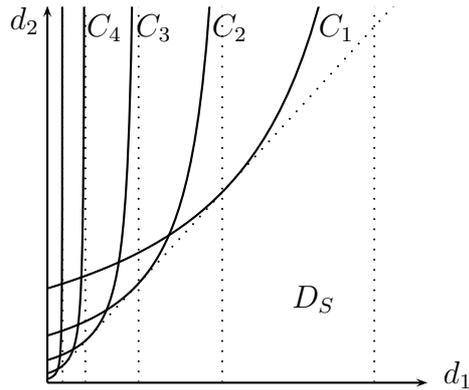


FIGURE 1. Hyperbolas (2.5) determining D_S , their vertical asymptotes, and the common tangential line

it is physically, is the cause of bifurcation of stationary nontrivial solutions (spatially nonconstant patterns) for (d_1, d_2) in D_S , *even if (d_1, d_2) lies under the common tangent of the hyperbolas C_n , even more, even if the quotient d_2/d_1 is extremely small!*

In fact, there is also a corresponding instability result in preparation which, however, we omit in these notes for brevity.

Let us make more precise what we mean by “forced positivity”/“obstacles” (for v) in this context. For simplicity, we describe only the case that there is a source on some nonempty (of positive $(N - 1)$ -Hausdorff measure) part Γ of the boundary $\partial\Omega$ (although the result holds also for the case of a sink or of a source in the interior). We mean by this that we replace (2.2) by so-called Neumann-Signorini boundary conditions

$$(2.6) \quad \begin{cases} \frac{\partial u}{\partial n} = 0 & \text{on } \partial\Omega, \\ v \geq 0, \quad \frac{\partial v}{\partial n} \geq 0, \quad \frac{\partial v}{\partial n} \cdot v = 0 & \text{on } \Gamma, \\ \frac{\partial v}{\partial n} = 0 & \text{on } (\partial\Omega) \setminus \Gamma. \end{cases}$$

A biochemical interpretation of the second line in (2.6) means that for those x with $v(x) > 0$ the Neumann condition (2.2) holds, but if $v(x)$ would become negative, a source ($\frac{\partial v}{\partial n} \geq 0$) becomes active (e.g. some cell on Γ produces the corresponding substance) which forces $v(x)$ to be 0.

Mathematically, we work in the Hilbert space $\mathbb{H} := \mathbb{H}_0 \times \mathbb{H}_0$ with $\mathbb{H}_0 := W^{1,2}(\Omega)$ being equipped with the usual scalar product

$$\langle u, \varphi \rangle := \int_{\Omega} u\varphi \, dx + \int_{\Omega} \nabla u \cdot \nabla \varphi \, dx,$$

define a symmetric compact positive definite operator $A_0: \mathbb{H}_0 \times \mathbb{H}_0$ by the linear equality

$$\langle A_0 u, \varphi \rangle = \int_{\Omega} u\varphi \, dx$$

(roughly speaking, this means that A_0 is a weak form of the operator $(id - \Delta)^{-1}$ under Neumann boundary conditions), and we define $A(d_1, d_2): \mathbb{H} \rightarrow \mathbb{H}$ by

$$A(d_1, d_2) \begin{pmatrix} u \\ v \end{pmatrix} := \begin{pmatrix} A_0 u \\ A_0 v \end{pmatrix} + \begin{pmatrix} \frac{1}{d_1}(b_{11}A_0 u + b_{12}A_0 v) \\ \frac{1}{d_2}(b_{21}A_0 u + b_{22}A_0 v) \end{pmatrix}.$$

Then a weak formulation of problem (2.4), (2.2) is the equation $U = A(d_1, d_2)U$ with $U = (u, v)$; as mentioned above, this has a nontrivial solution if and only if $(d_1, d_2) \in C_n$ for some $n \geq 1$.

For a weak formulation with the obstacles, we consider the cone

$$K := \left\{ \begin{pmatrix} u \\ v \end{pmatrix} \in \mathbb{H} : v|_\Gamma \geq 0 \right\},$$

where the positivity condition $v|_\Gamma \geq 0$ means that the corresponding trace is almost everywhere nonnegative on Γ . Then a weak formulation of problem (2.4), (2.6) is given by the variational inequality (with $U = (u, v)$)

$$U \in K, \quad \langle U - A(d_1, d_2)U, \Phi - U \rangle \geq 0 \text{ for all } \Phi \in K.$$

As mentioned in the previous section this can be equivalently rewritten as

$$U = P_K(A(d_1, d_2)U).$$

The main result in [8] states that there are constants $\omega_1, \omega_2, d_0 > 0$ and for every $\varepsilon \in (0, d_0)$ some $\omega_\varepsilon > 0$ such that there is a (relatively closed) connected set $\mathcal{C} \subseteq D_S$ which “separates” the shaded unbounded “squares” D_\pm sketched in Figure 2 such that (2.4), (2.6) has a nontrivial solution (“critical point”) for every $(d_1, d_2) \in \mathcal{C}$.

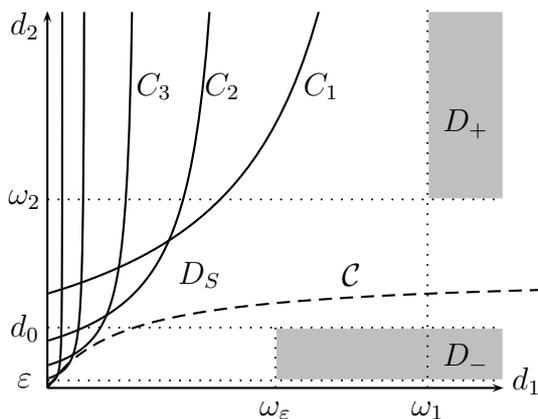


FIGURE 2. Bifurcation points of (2.7), (2.6) and the two zones D_\pm

Moreover, the points from \mathcal{C} are automatically bifurcation points of the nonlinear stationary problem

$$(2.7) \quad \begin{aligned} 0 &= d_1 \Delta u + f_1(u, v), \\ 0 &= d_2 \Delta v + f_2(u, v) \end{aligned}$$

with obstacles (2.6), and the bifurcation is global in the sense mentioned earlier along every path meeting D_+ and D_- , see [8] for details. In dimension $N = 1$, the set \mathcal{C} was “explicitly calculated” and discussed in [4] (with the result that Figure 2 is realistic).

As mentioned in the beginning, there is a result in preparation [5] concerning instability: In order to keep these notes short, we just mention that this result implies that $(0, 0)$ fails to be asymptotically stable in (2.1), (2.6) if $(d_1, d_2) \in D_S$ lies “above” all critical points from D_S (that is, roughly speaking, in the area between \mathcal{C} and the envelope of

the hyperbolas C_n). It is somewhat surprising that this instability result is obtained by degree methods.

The above mentioned bifurcation result solves (implicit) conjectures from [2] which were open for almost 30 years. In the case that on some nonempty (of positive $(N - 1)$ -measure) part Γ_0 of the boundary $\partial\Omega$ with $\bar{\Gamma}_0 \cap \bar{\Gamma} = \emptyset$ there is a Dirichlet condition prescribed for u and v , i.e., if (2.6) is replaced by the Dirichlet-Neumann-Signorini conditions

$$\begin{cases} u = v = 0 & \text{on } \Gamma_0, \\ \frac{\partial u}{\partial n} = 0 & \text{on } \partial\Omega \setminus \Gamma_0, \\ v \geq 0, \quad \frac{\partial v}{\partial n} \geq 0, \quad \frac{\partial v}{\partial n} \cdot v = 0 & \text{on } \Gamma, \\ \frac{\partial v}{\partial n} = 0 & \text{on } (\partial\Omega) \setminus (\Gamma_0 \cup \Gamma), \end{cases}$$

the corresponding problem is much easier and was essentially already solved in [2] (see also e.g. [1, 10] or [14]): In this case, one works in the space

$$\mathbb{H}_0 := \{u \in W^{1,2}(\Omega) : u|_{\Gamma_0} = 0\}$$

which is equipped with the equivalent scalar product

$$\langle u, \varphi \rangle := \int_{\Omega} \nabla u \cdot \nabla \varphi \, dx.$$

In this case, one lets $0 < \kappa_1 < \kappa_2 < \dots \rightarrow \infty$ denote the eigenvalues of $(-\Delta)$ with mixed Dirichlet-Neumann condition

$$(2.8) \quad u|_{\Gamma_0} = 0 \quad \text{and} \quad \frac{\partial u}{\partial n} \Big|_{(\partial\Omega) \setminus \Gamma_0} = 0.$$

The reason why this problem is so much easier than the problem without Dirichlet conditions is that due to the simpler scalar product in \mathbb{H}_0 the corresponding operator A_0 corresponds to the weak form of $(-\Delta)^{-1}$ (with boundary conditions (2.8)), and so the operator $A(d_1, d_2)$ appearing in the weak formulation becomes in this case

$$A(d_1, d_2) \begin{pmatrix} u \\ v \end{pmatrix} := \begin{pmatrix} \frac{1}{d_1}(b_{11}A_0u + b_{12}A_0v) \\ \frac{1}{d_2}(b_{21}A_0u + b_{22}A_0v) \end{pmatrix},$$

i.e. $A(d_1, d_2)$ depends linearly on (d_1^{-1}, d_2^{-1}) , and its eigenvalues are far away from 1 for large d_1 and d_2 (in the non-Dirichlet case the failure of these is the most severe technical difficulty: one cannot “factor out” the corresponding eigenspace as in linear problems.)

Without obstacles, the results mentioned earlier hold with Dirichlet conditions word by word, but one can calculate that the obstacles cannot induce any bifurcation point or critical value in the zone

$$(2.9) \quad Z_0 := \left(\frac{b_{11}}{\kappa_1}, \infty\right) \times (0, \infty),$$

that is, to the right of the asymptote of the right-most hyperbola. However, there is a branch of bifurcation points in D_S nevertheless, see e.g. Figure 3.

Thus, we have a qualitatively completely different picture in the Dirichlet case than in the non-Dirichlet case.

In case of obstacles with the Dirichlet conditions much more is known (as mentioned earlier, the Dirichlet case is much simpler): For instance, the existence of certain branches of bifurcation points “between” the hyperbolas C_n is known [14], and it has been studied what happens if there are simultaneous obstacles for u and v [3].

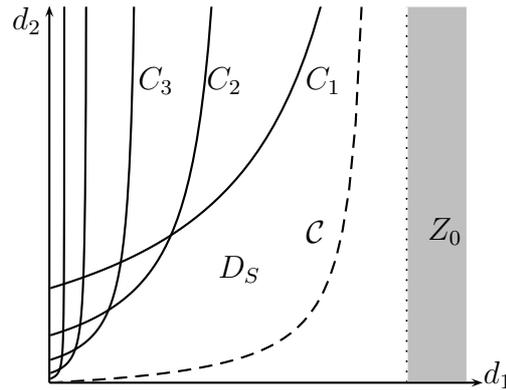


FIGURE 3. Bifurcation points in the Dirichlet case and the zone (2.9)

REFERENCES

- [1] Drábek, P. and Kučera, M., *Reaction-diffusion systems: Destabilizing effect of unilateral conditions*, *Nonlinear Anal.* **12** (1988), no. 11, 1172–1192.
- [2] Drábek, P., Kučera, M., and Míková, M., *Bifurcation points of reaction-diffusion systems with unilateral conditions*, *Czechoslovak Math. J.* **35** (1985), 639–660.
- [3] Eisner, J., Kučera, M., and Váth, M., *Bifurcation points for a reaction-diffusion system with two inequalities*, *J. Math. Anal. Appl.* **365** (2010), 176–194.
- [4] Eisner, J. and Váth, M., *Location of bifurcation points for a reaction-diffusion system with Neumann-Signorini conditions*, *Advanced Nonlinear Studies* **11** (2011), 809–836.
- [5] Kim, I.-S. and Váth, M., *The Krasnosel'skiĭ-Quittner formula and instability of a reaction-diffusion system with unilateral obstacles*, (in preparation).
- [6] Kučera, M., *Bifurcation points of variational inequalities*, *Czechoslovak Math. J.* **32** (1982), 208–226.
- [7] ———, *A new method for obtaining eigenvalues of variational inequalities. Multiple eigenvalues*, *Czechoslovak Math. J.* **32** (1982), 197–207.
- [8] Kučera, M. and Váth, M., *Bifurcation for a reaction-diffusion system with unilateral and Neumann boundary conditions*, *J. Differential Equations* **252** (2012), 2951–2982.
- [9] Quittner, P., *Spectral analysis of variational inequalities*, *Comment. Math. Univ. Carolinae* **27** (1986), no. 3, 605–629.
- [10] ———, *Bifurcation points and eigenvalues of inequalities of reaction-diffusion type*, *J. Reine Angew. Math.* **380** (1987), no. 2, 1–13.
- [11] ———, *Solvability and multiplicity results for variational inequalities*, *Comment. Math. Univ. Carolinae* **30** (1989), no. 2, 281–302.
- [12] Rabinowitz, P. H., *Some global results for nonlinear eigenvalue problems*, *J. Funct. Anal.* **7** (1971), 487–513.
- [13] Turing, A. M., *The chemical basis of morphogenesis*, *Phil. Trans. R. Soc. London Ser. B* **237** (1952), no. 641, 37–72.
- [14] Váth, M., *New beams of global bifurcation points for a reaction-diffusion system with inequalities or inclusions*, *J. Differential Equations* **247** (2009), 3040–3069.
- [15] ———, *Calculation of degree and global bifurcation for variational inequalities with nonsymmetric operators in Banach spaces*, *Ann. Mat. Pura Appl.* (2012), 1–25.

FREE UNIVERSITY OF BERLIN, MATH. INF., ARNIMALLEE 3, D-14195 BERLIN, GERMANY
E-mail address: vaeth@mathematik.uni-wuerzburg.de